Irrational Trilogy
This Welcome is brought to you by Infinite Descent, the Rational Root Theorem, and the Irrational Number $\sqrt{2}$

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

The Pythagorean: Hippasus of Metapontum

> Hippasus (c. 500 BCE) is sometimes credited with the discovery of irrational numbers. He was a member of the Pythagoreans, a secretive group. Perhaps he was drowned for revealing the existence of irrationals. A similar story is told the revealer of the existence of the dodecahedron, so maybe it is not true.

It never hurts to introduce a little danger into a math talk. There is evidence, that the golden ratio was the first number found to be irrational, and not $\sqrt{2}$.

Infinite Descent, Escher's Waterfall

We are going to get a contradiction by constructing an infinite strictly decreasing sequence of positive integers. Something that cannot happen. This proof is sometimes described as infinite descent. Escher's Waterfall is a good image for this idea.

Suppose $\sqrt{2} = a/b$ in lowest terms

Let $a_n = (\sqrt{2}-1)^n a$ and $b_n = (\sqrt{2}-1)^n b$. Since $1 < \sqrt{2} < 2$, $0 < \sqrt{2} - 1 < 1$ and

$$a_0 > a_1 > a_2 > \cdots > 0, \qquad b_0 > b_1 > b_2 > \cdots > 0$$

$$\frac{a_n}{b_n} = \frac{a}{b} = \sqrt{2} \implies \sqrt{2} b_n = a_n \qquad \sqrt{2} a_n = 2b_n$$

%pause

$$a_{n+1} = (\sqrt{2}-1)a_n = 2b_n - a_n, \qquad b_{n+1} = (\sqrt{2}-1)b_n = a_n - b_n$$

Both sequences are strictly decreasing sequences of positive integers.

First we define the integer sequences a_n and b_n. Second we note that we are multiplying by a positive number less than one so the sequences are strictly deceasing. We use $a_n/b_n = \sqrt{2}$ to obtain

$$a_{n+1} = 2b_n - a_n \text{ and } b_{n+1} = a_n - b_n$$

which shows if a_n and b_n are integers so are a_{n+1} and b_{n+1}. The infinite descent is complete and $\sqrt{2}$ is irrational.

This is not far from the usual proof. See a geometric version http://www.cut-the-knot.org/proofs/sq_root.shtml#proof7 and 28 other proofs.

It Ain't Me Babe

The next proof shows that each rational a/b is relatively far from $\sqrt{2}$. "It Ain't Me Babe" is a Dylan song, but it is on a different album.

For integers a and b, $|\sqrt{2} - a/b| > 1/3b^2$

Case 1: $0 \le a/b \le 3/2$,

$$|\sqrt{2} - a/b| = |\sqrt{2} - a/b| \frac{|\sqrt{2} + a/b|}{|\sqrt{2} + a/b|} = \frac{|2 - a^2/b^2|}{\sqrt{2} + a/b} > \frac{|2b^2 - a^2|}{3b^2} \ge \frac{1}{3b^2}$$

Case 2: $a/b > 3/2$, $|\sqrt{2} - a/b| > |3/2 - a/b| = |3b - 2a|/|2b| > 1/3b^2$

Case 3: $a/b < 0$,$|\sqrt{2} - a/b| > 1 > 1/3b^2$

For Case 1; we start by rationalizing the numerator. We decrease the fraction by increasing the bottom from $\sqrt{2} + a/b$ to the larger 3 and we decrease the top by noting $|2b^2 - a^2|$ is an integer, which is non-zero and hence greater or equal to one. In Case 2, replace $\sqrt{2}$ with the larger 3/2. Again $|3b - 2a|$ is an non-zero integer and we decrease the fraction by increasing the bottom from $|2b|$ to the larger $3b^2$. Case 3, the number is larger than $\sqrt{2}$ and certainly $1/3b^2$.

One needs to know $\sqrt{2}$ is irrational in this proof, so that $2b^2 - a^2 \neq 0$. Algebra shows

$$2b^2 - a^2 = 0 \iff (a/b)^2 = 2.$$

Using convergents of continued fractions, it is known that there are infinitely many fractions a/b so that

$$|\sqrt{2} - a/b| < 1/b^2.$$

Precalculus Theorem about Rational Roots

Theorem. If p and q are relatively prime and p/q is a root of $a_n x^n + a_{n-1}x^{n-1} + \ldots a_1 x + a_0$ with integer coefficients then $p \mid a_0$ and $q \mid a_n$.

Proof: Substitute $x = p/q$ and multiply by q^n

$$\overbrace{a_n p^n + \underbrace{a_{n-1}p^{n-1}q + \cdots + a_1 pq^{n-1}}_{\text{divisible by } p} + a_0 q^n}^{\text{divisible by } q} = 0$$

This is the rational root theorem. Since $p \mid 0$, the right hand side of the equation and the first $n-1$ terms, it must divide the last term $p \mid a_0 q^n$. But p and q are relatively prime so $p \mid a_0$. The $q \mid a_n$ case is similar.

My Calculus 1 instructor started our class proving this theorem. The theorem was not in my calculus book. All of the teaching assistants at Florida State University usually have Precalculus as their first solo class, and Calculus I as their second.

Dead On or Completely Off Base

Corollary: A root of monic polynomial is either an integer or an irrational.

Because $q \mid 1$ it, follows $q = \pm 1$.

Since 2 is not a square, $\sqrt{2}$ is a non-integer root of the monic polynomial $x^2 - 2$ and hence is irrational.

A polynomial is called monic if the coefficients are integers and the leading coefficient is one. Since the bottom $q = \pm 1$ any rational roots are integers. We have a direct proof, that $\sqrt{2}$, an non-integer root of the monic $x^2 - 2$, is irrational.

Picture sources

1. Hippasus picture is from `https://pointatinfinityblog.wordpress.com/2018/05/07/hippasus-and-the-infinite-descent/`.

2. Escher picture is from `https://en.wikipedia.org/wiki/Waterfall_(M._C._Escher)`

4. Dylan picture is from the album cover in iTunes.