The Good, The Bad and The Ugly

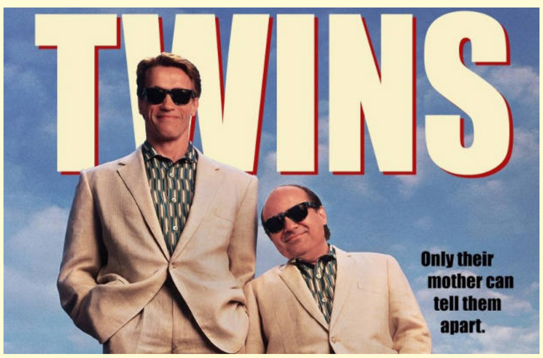
This welcome is brought to you
by Euler,
Borel, and
Peano

Steven F. Bellenot

Jan 3, 2014

The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Students: The Good and the Bad.



The 1988 Movie: Twins

A physically perfect but innocent man goes in search of his long-lost twin brother, who is a short small-time crook.

There is a sequel planned called Triplets, adding Eddie Murphy.

The good, the bad and the ugly is a 1966 Clint Eastwood Movie.

That is what I said.

new stuff

Analysis: The Real and the Complex

In Real Analysis counter-examples rule, there are few univerisal theorems.

In Complex Analysis universal theorems rule, there are few counter-examples.

The sequel?: the Complex, the Real and the Numerical.

That is what I said.

new stuff

$g(x)$, a function considered by Euler

$$g(x) = \int_0^\infty \frac{e^{-t}}{1+xt}\,dt$$

If $x < y$, then $1/(1+xt) > 1/(1+yt)$ so $g(x) > g(y)$ and g is monotone decreasing. Eventually all derivatives of $g(x)$ are monotone. (Alternating between decreasing and increasing.)

It is a nice function, $g(x)$ is C^∞ for $x \in [0, \infty)$ and

$$g^{(n)}(0) = (-1)^n (n!)^2$$

The Taylor series for $g(x) = \sum (-1)^n n! x^n$ which diverges for all non-zero x.

Using Cauchy principle values, $g(x)$ is also defined for $x < 0$.

That is what I said.

new stuff

A Lemma

By induction, using integration by parts:

$$\int_0^\infty t^n e^{-t}\,dt = n!$$

Let $u = t^{n+1}$ and $dv = e^{-t}$

$$\int_0^\infty t^{n+1} e^{-t}\,dt = -e^{-t} t^{n+1}\big|_0^\infty - \int_0^\infty -e^{-t}(n+1)t^n\,dt$$

$$= 0 + (n+1)n!$$

That is what I said.

new stuff

Derivatives of $g(x)$

If $h(x) = \int_a^b f(x,t)\,dt$, then

$$h'(x) = \int_a^b \frac{\partial f(x,t)}{\partial x}\,dt$$

$$g^{(n)}(x) = \int_0^\infty \frac{(-1)^n n! t^n e^{-t}}{(1+xt)^{n+1}}\,dt$$

$$g^{(n+1)}(x) = \int_0^\infty \frac{(-1)^{n+1}(n+1)!t^{n+1}e^{-t}}{(1+xt)^{n+2}}\,dt$$

$$g^{(n)}(0) = \int_0^\infty (-1)^n n! t^n e^{-t}\,dt = (-1)^n (n!)^2$$

That is what I said.

new stuff

$g(z)$, a multi-valued complex function

$$g(z) = \int_0^\infty \frac{e^{-t}}{1+zt}\,dt$$

If the closed curve $C \subset \mathbb{C}\backslash(-\infty, 0]$, then $\int_C g(z)\,dz = 0$ so $g(z)$ is analytic with a branch cut at $(-\infty, 0]$

$$\int_C g(z)\,dz = \int_C \int_0^\infty \frac{e^{-t}}{1+zt}\,dt\,dz$$
$$= \int_0^\infty \int_C \frac{e^{-t}}{1+zt}\,dz\,dt = \int_0^\infty 0\,dt = 0$$

$g(z)$ has ∞-many values like and similar to $\log(z)$.

That is what I said.

new stuff

Borel's Theorem

For any sequence of reals $\{a_n\}$ there is a C^∞ function $f(x)$ whose Taylor's series at $x = 0$ is $\sum a_n x^n$.

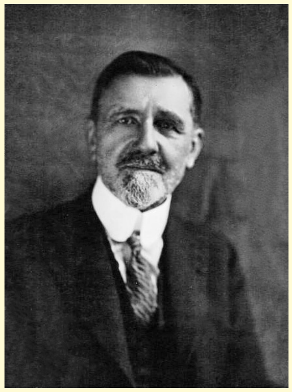
This is not true for functions analytic at $x = 0$, no analytic function has Taylor series $\sum (-1)^n n! x^n$.

Borel's Theorem was proved by Peano over decade before Borel proved it.

That is what I said.

new stuff

Borel and Peano

That is what I said.

new stuff

Picture sources

x. picture is from