Student partial classification

This welcome is brought to you
by d'Alembert,
Fourier,
and Laplace

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Paul du Bois-Reymond (1889) PDE Classification

Bois-Reymond (1831-1889) was the first to classify two dimensional Partial Differential Equations (PDE) as elliptic, parabolic or hyperbolic in 1889.

Classifications of 2-D PDE's

Write $Au_{xx}+2Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$ as the quadratic $Ax^2+2Bxy+Cy^2+Dx+Ey+F=0$ and classify by the eigenvalues of quadratic form

$$\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

opposite signs means hyperbolic, same signs means elliptic, one zero parabolic

The classification starts with converting the PDE into a quadratic in two variables. We look at the associated quadratic form, which is a symmetric matrix hence diagonlizable. Basically the classification matchs the classification of the quadratic into ellipses, hyperbolas and parabolas.

Pierre-Simon Laplace (c1780) Laplace's Equation

Laplace (1749-1827) solved what is now called Laplace's Equation in 1780 at age 31.

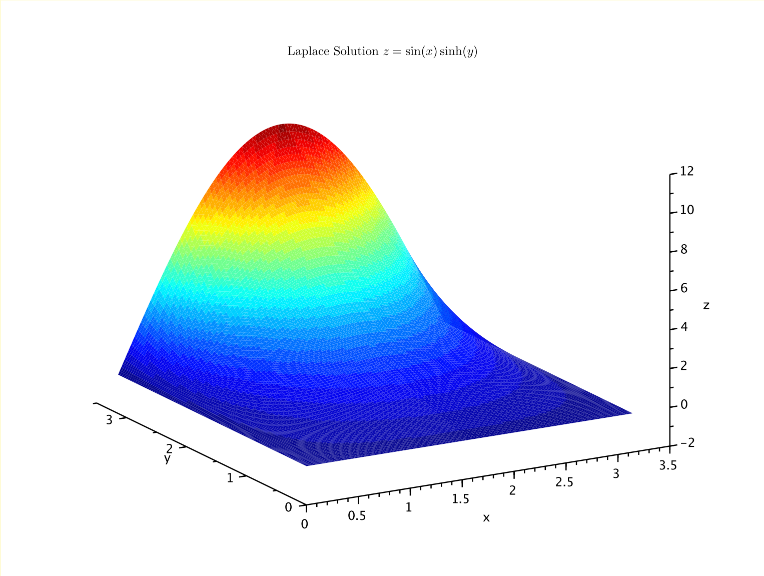
Properties of Solutions

$$u_{xx}+u_{yy}=0$$

Speed Zero
Solutions Harmonic – Power Series
Elliptic
Steady State Temperature

Elliptic equations are about steady state where nothing changes, hence the zero speed. Their solutions are a smooth as possible being power series. A classic model is a temperture in steady state.

Typical Solution



This function $\sin x \sinh y$ is a solution inside the region if three edges are kept at zero, and the fourth is kept at $\sin x$.

Jean le Rend d'Alembert (1748) Wave Equation

d'Alembert (1717-1783) solved the wave equation in 1748 at age 31.

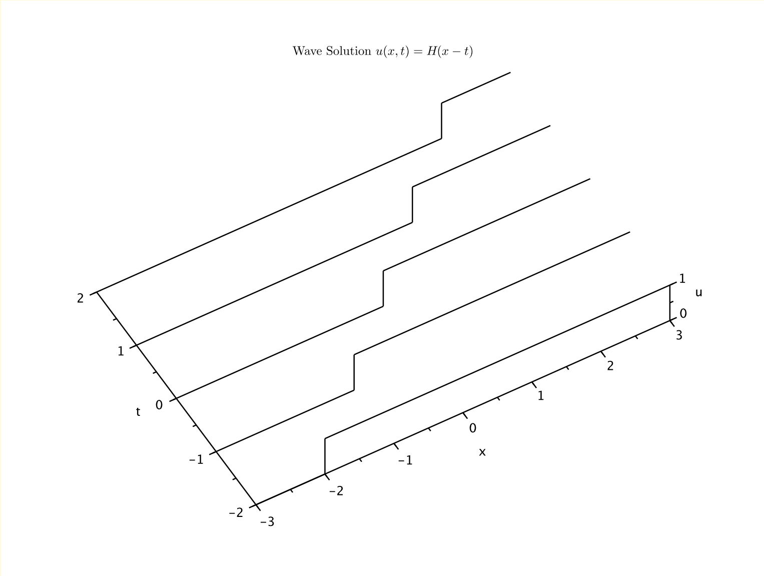
Properties of Solutions

$$u_{xx} = u_{tt}$$

Finite Speed
Discontinuous Solutions
Hyperbolic
Vibrating String

The solutions of the wave equation has waves traveling at finite speed and their solution are far from smooth, they can be discontinuous. A classic model is a vibrating string.

Typical Solution



Here a discontinuous Wave solution $u(x,t) = H(x-t)$ which we will look at in more detail later. The wave of discontinuity is moving from left to right.

Jean-Baptiste Fourier (1810-1822) Heat Equation

Fourier (1768-1830) solved the heat equati at the age of 42. Because of his use of what is now called Fourier series, there were questions. His solution was not immediately accepted. He is also famous for discovering the green house effect and for dimensional analysis. He is one of my mathematical great great great great great great grand fathers.

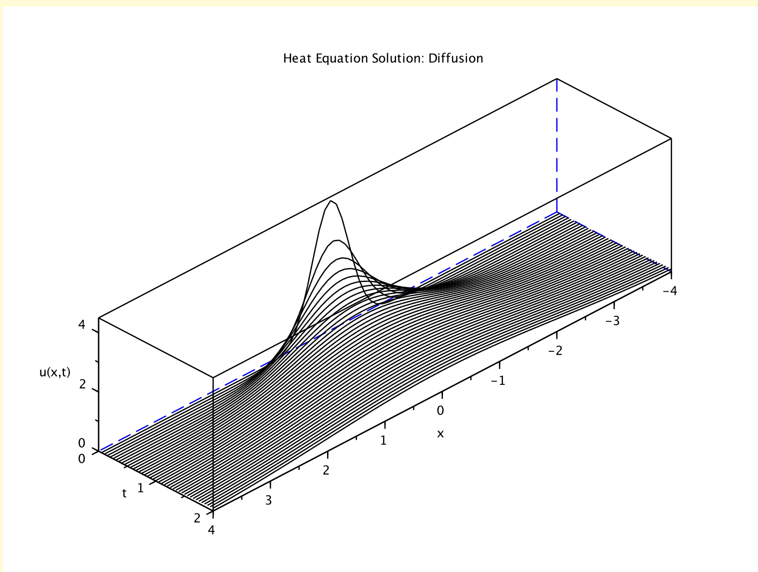
Properties of Solutions

$$u_{xx} = u_t$$

Speed Infinity
Solutions C^∞
Parabolic
Diffusion

The classic model is diffusion. The solutions are C^∞ so they have continuous derivatives of all orders. Diffusion happens infinity fast. A square wave for example immediately is non-zero everywhere.

Typical Solution



Here we see the Gaussian shaped hill at time zero, smear out as time increases.

$u = H(x-t)$ is a solution to $u_{xx} - u_{tt} = 0$

H is the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 1/2 & x = 0 \\ 0 & x < 0 \end{cases}$$

Show $u_x + u_t = 0$

We are going to show that the Heaviside discontinuous step function is a to the wave equation. Since discontinuous functions are not differentiable, we show it is a weak solution.

The inner product

$$\langle f(x,t), g(x,t)\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,t)g(x,t)\,dxdt$$

is used. Since g will alway have compact support, the integral is really over a bounded rectangle. We will use ϕ as a typical test function which is both C^∞ and all its derivatives have compact support.

Step 1: Integrate with respect to x first

$$\begin{aligned}
\langle u_x, \phi\rangle &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\phi_x(x,t)\,dx\,dt \\
&= -\int_{-\infty}^{\infty}\int_{t}^{\infty} \phi_x(x,t)\,dx\,dt \\
&= -\int_{-\infty}^{\infty} -\phi(t,t)\,dt \\
&= \int_{-\infty}^{\infty} \phi(t,t)\,dt
\end{aligned}$$

The inner product is the integral, we use integration by parts to move the derivative with respect to x over to the C^∞ function ϕ. Since $u(x,t) = 0$ for $x < t$, we get the second line. The fundamental theorem of calculus, yields the third line. Finally two negatives make a positive.

Integration by parts

$$\int u\,dv = uv - \int v\,du$$

is used with $dv = u_x$ and $u = \phi$

$$\begin{aligned}
\int_{-\infty}^{\infty} u_x\phi\,dx &= u\phi|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u\phi_x\,dx \\
&= -\int_{-\infty}^{\infty} u\phi_x\,dx
\end{aligned}$$

Since $u\phi$ is zero for points far away from zero.

Step 2: Integrate with respect to t first

$$\begin{aligned}\langle u_t, \phi\rangle &= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\phi_t(x,t)\,dt\,dx \\ &= -\int_{-\infty}^{\infty}\int_{-\infty}^{x} \phi_t(x,t)\,dt\,dx \\ &= -\int_{-\infty}^{\infty} \phi(x,x)\,dx \\ &= -\int_{-\infty}^{\infty} \phi(t,t)\,dt\end{aligned}$$

Again, integration by parts moves the derivative with respect to t from u to ϕ. Since $u(x,t) = 0$ for $x < t$, we get the second line. The fundamental theorem of calculus yields the third line. We change the dummy variable, to get the last line.

Step 3

We now know $u_x + u_t = 0$ so

$$0 = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial t}\right)(u_x + u_t) = u_{xx} + u_{xt} - u_{tx} - u_{tt} = u_{xx} - u_{tt}$$

We have shown $u_x + u_t = 0$ weakly. Using the operator $\partial_x - \partial_t$ gives the wanted result.

Picture sources

d'Alembert picture is from Wikipedia
https://en.wikipedia.org/wiki/D%27Alembert%27s_paradox

Bois-Reymond picture is from Wikipedia
https://en.wikipedia.org/wiki/Paul_du_Bois-Reymond

Fourier picture is from Wikipedia
https://en.wikipedia.org/wiki/Joseph_Fourier

Laplace picture from
https://famous-mathematicians.com/pierre-simon-laplace/

Plots Were made by the author using Scilab.