Something odd, something prime and gamma too

This welcome is brought to you
by the contant gamma γ,
the harmonic series, and
Bertrand's postulate

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

new stuff

The constant γ

γ the Euler-Mascheroni constant

$$\gamma = 0.57723\ 56649\ 01532\ 86060\ 65120\ldots$$

$$\gamma = \lim_n \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right)$$

$$\Gamma'(1) = -\gamma$$

If $\gamma = p/q$ then $q > 10^{242080}$

Euler has two constants named after him, everyone knows e, the base of natural logarithms. This is one lower case gamma γ, which only loosely connected to the upper case Gamma function $\Gamma(z)$. It is defined as the limit shown. Latter we will see a geometric picture of the limit. It is not known if γ is rational. If it is rational the denominator must be large, at least quarter million digits.

Something odd

The harmonic series

$$H_n = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n > 1$.

Proof: Let 2^j be the largest power of two in $1 \dots n$ and let $D = \text{lcm}\{1 \dots n\}$. Note

$$\frac{1}{2^j} = \frac{N_0}{D} \text{ with } N_0 \text{ odd.}$$

$$\frac{1}{k} = \frac{N_k}{D} \text{ with } N_k \text{ even otherwise.}$$

$$\therefore H_n = \frac{N}{D} \text{ with } N \text{ odd and } D \text{ even.}$$

The harmonic series, one of our favorites is known to diverge getting bigger than any integer. It steps over every positive integer larger than one.

Something Prime

Bertrand's postulate: There is always a prime number between k and $2k$.

Erös elementary proof: If there was no such prime, then the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$$

would be too small.

That is what I said.

Non terminating rationals

H_n is a non-terminating rational for $n > 6$.
There is a prime $p \geq 7$ with $n/2 < p \leq n$. Let D be the lcm as before

$$\frac{1}{p} = \frac{N_0}{D}, \text{ with } N_0 \ncong 0 \mod p$$

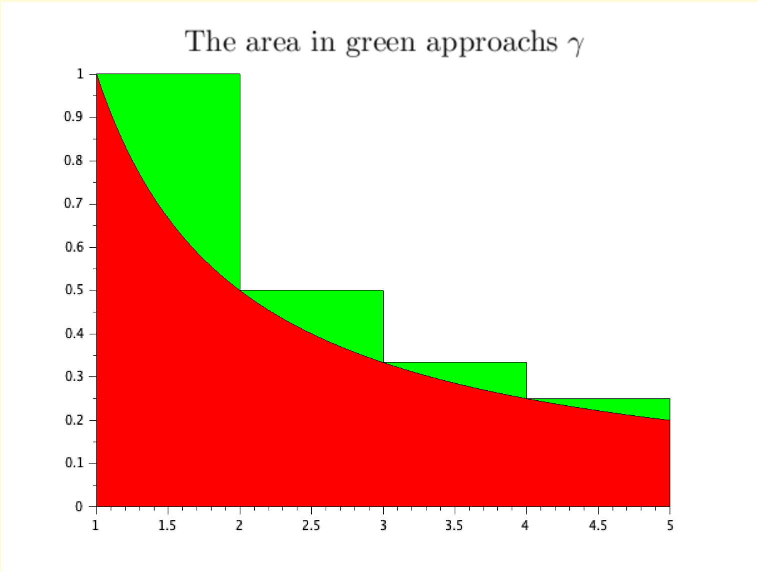
$$\frac{1}{k} = \frac{N_k}{D}, \text{ with } N_k \cong 0 \mod p$$

$$\therefore H_n = \frac{N}{D}, \text{ with } N \ncong 0 \mod p$$

$$H_6 = \frac{147}{60} = \frac{49}{20} = 2.45$$

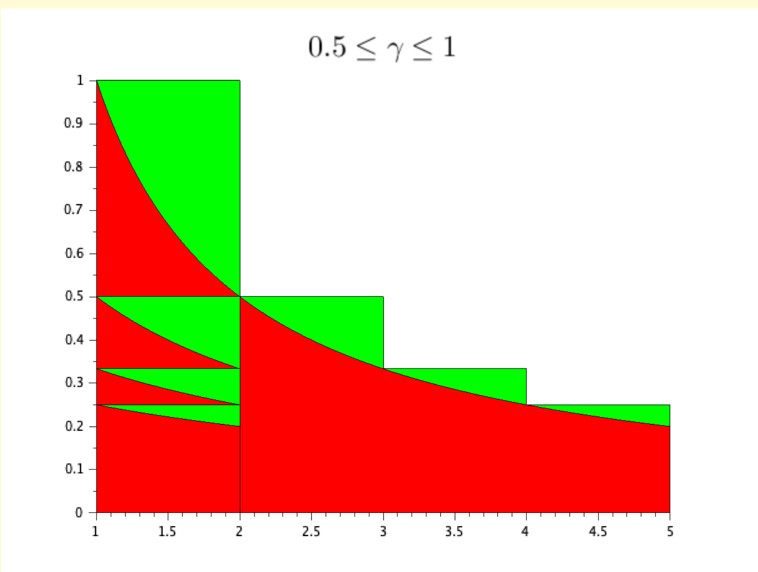
That is what I said.

γ Geometrically



That is what I said.

γ Geometrically



That is what I said.

Taylor series for $\ln(1+x)$

$$\frac{d}{dx}(\ln(1+x)) = \left(\frac{1}{1+x}\right) = \sum(-1)^n x^n$$
$$\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$
$$\ln(1+\frac{1}{r}) = \frac{1}{r} + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{1}{nr^n}$$
$$\frac{1}{r} = \ln(1+\frac{1}{r}) + \sum_{n=2}^{\infty}(-1)^n \frac{1}{nr^n}$$

That is what I said.

Telescoping Logs

$$\ln\left(1+\frac{1}{r}\right) = \ln\left(\frac{r+1}{r}\right)$$
$$= \ln(r+1) - \ln(r)$$
$$\sum_{r=1}^{n}\ln(1+1/r) = \ln(n+1) - \ln(1) = \ln(n+1)$$

That is what I said.

Punch line

$$\sum_{r=1}^{n}\frac{1}{r} = \ln(n+1) + \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r^2} - \frac{1}{3}\sum_{r=1}^{n}\frac{1}{r^3} + \dots$$
$$\sum_{r=1}^{n}\frac{1}{r} - \ln(n+1) = \sum_{k=2}^{\infty}(-1)^k\frac{1}{k}\sum_{r=1}^{n}\frac{1}{r^k}$$

The Alternating Series test gives an estimate on γ.

That is what I said.

Picture sources

5 Graph by author

6 Graph by author