Space Filling

This welcome is brought to you
by Hilbert's curve,
Measure Zero,
and the extra room in three dimensions

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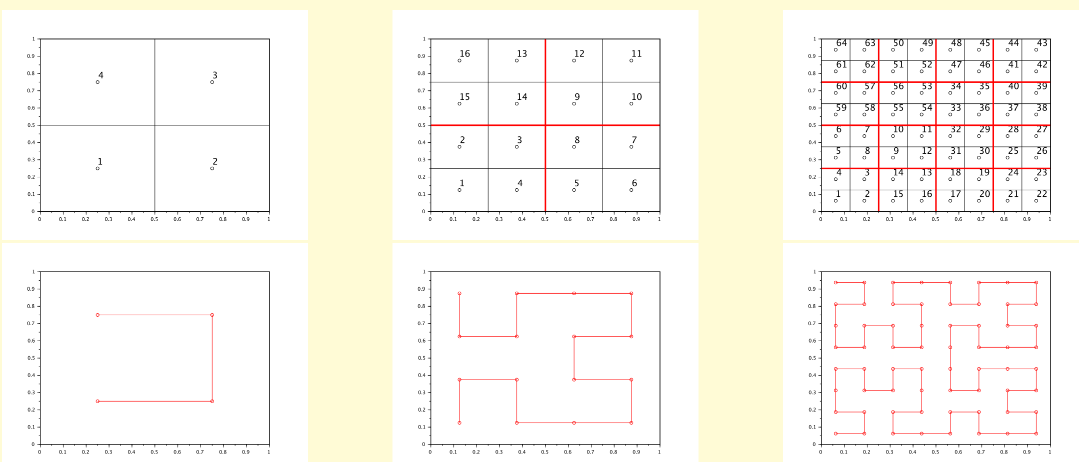
The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Hilbert's space filling curve

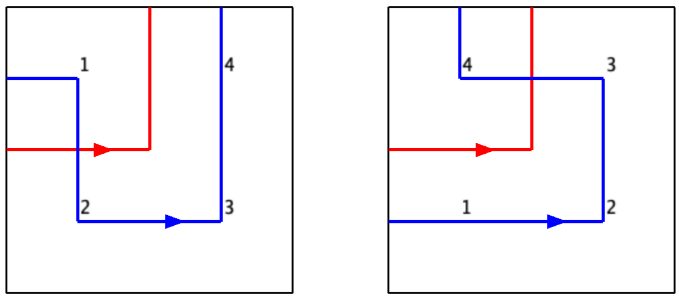


This slide shows how to construct Hilbert's space filling curve. The top row shows the trick, to inductively subdivide each square into four subsquare in a manner that allows a path to go throught the numbered squares linearly. The portion of the path in any square, is redrawn so it spends 1/4 of the time in the square, in each subsquare.

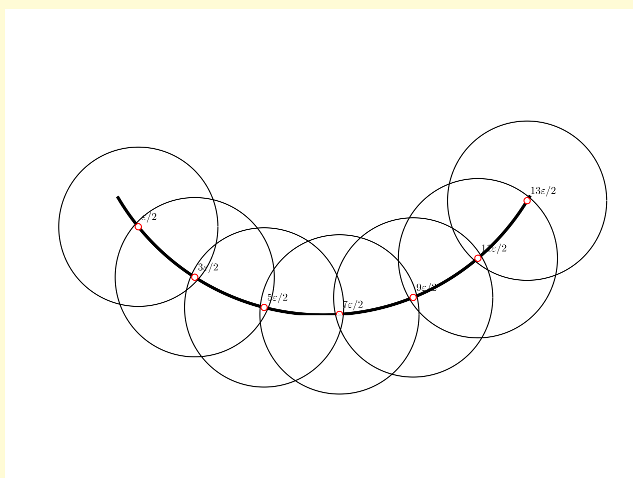
The curve is the limit of these piecewise linear paths shown on the bottom. The pattern is to connect adjacent centers in a fashion which preserves the time it is in these squares. So the period $0 \leq t \leq 1/4$ is always in the first big square, the period $0 \leq t \leq 1/16$ is always in the first square in the second picture and so on. Every point on the square is the limit point of the sequence of curves, since the point is in a nested sequence of ever smaller subsquares. Since the limit is uniform, the curve is continous.

Peano discovered the first space filling curve in 1890, a continous function from $[0, 1]$ to the unit square which was onto. The Hilbert curve has a nice geometry, an is from the same time period 1891.

The pair of graphs below show a detail with pleasing picture. The figures below show how to replace the old curve, in red, at one level, with a new curve, in blue at the next level and shows how to number the subdivided squares. There are two cases depending on which side of the old curve the new curve enters this square.



Rectifyable curves have measure zero



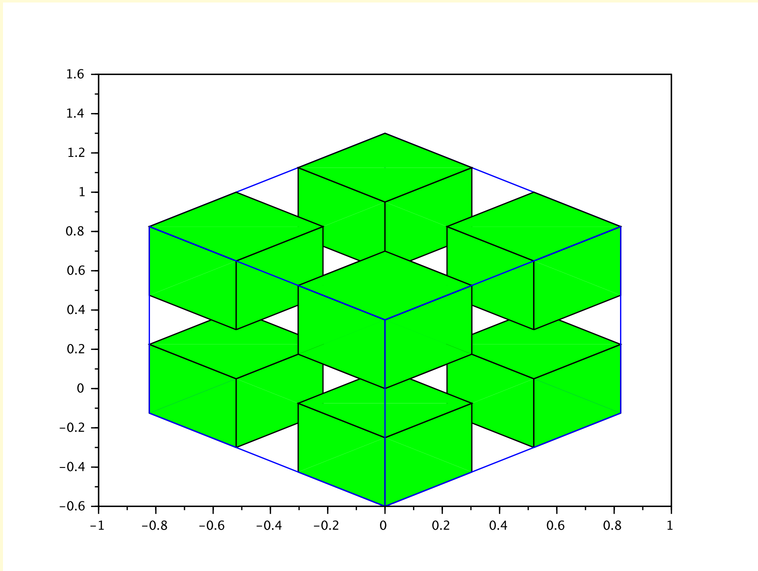
$$\frac{L}{\varepsilon}\pi\varepsilon^2$$

This picture shows any curve with length (i.e. rectifyable curve) has measure zero. So Hilbert's curve has infinite length.

For each ε we center circles of radius ε roughly ε apart along the curve. There are L/ε circles, so the area covered is $L\pi\varepsilon$. Which is small as we like.

Hence any curve with positive area is infinity long.

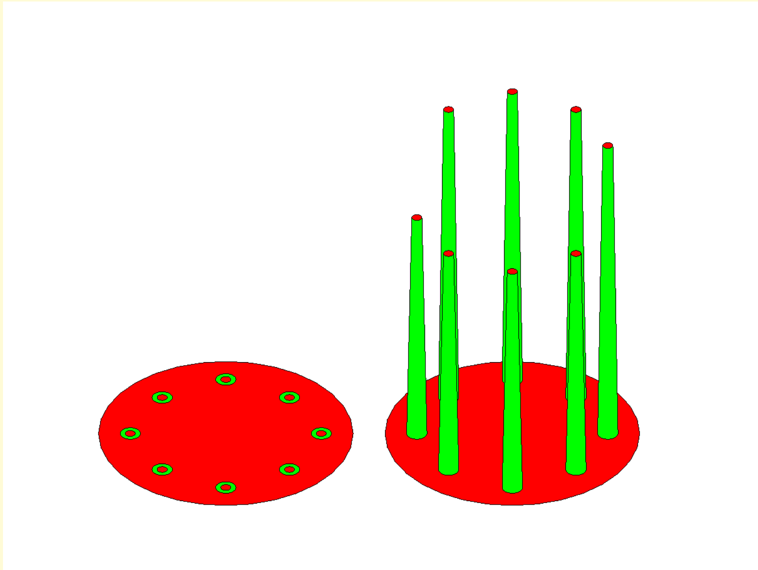
Volume $M^3 - 1$: $3 \cdot 2^n M^2 \delta_n < 1/2^n$



We begin a construction of a surface with large volume $M^3 - 1$ with very small surface area. To get the large volume we construct a generalized Cantor set with volume near M^3 by deleting three pieces middle pieces of thickness δ_n in each coordinate direction, creating 8 cubes. We repeat, and the infinite process creates a volume of Cantor dust with measure near M^3.

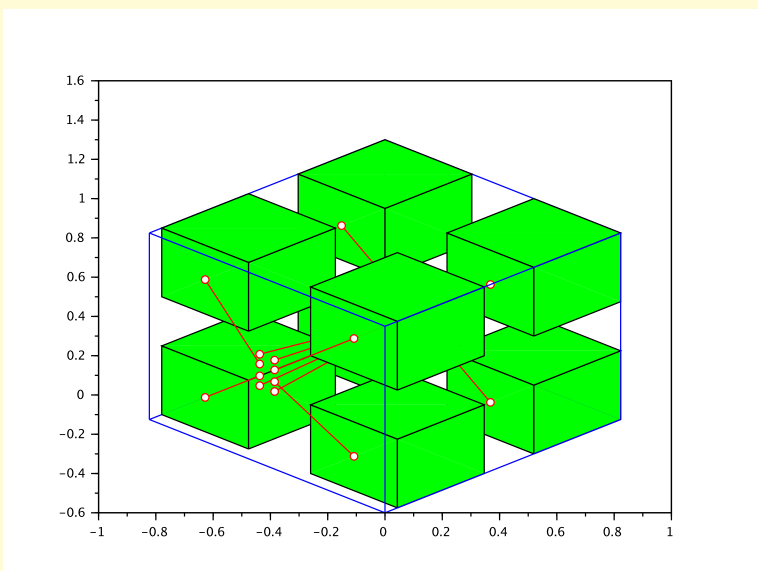
One could do this kind of construction inside the unit square, to get a Cantor dust with area as near one as you like.

Surface area ε: Surface Area, $8^n \gamma_n M/2^n < \varepsilon/2^n$



Next we construct a surface inductively so that its surface area is small. We add airconditioning ducts connecting the cubes from the previous construction. The length of each tube is less than the diagonal, so if we pick the new disk size as γ_n the total surface is small enougth that the whole surface area is small, less than ε.

Slightly shifted with added AC



Here is the aircondition ducts. Every point on the dust, is a limit of the disks created as we go along. So the surface will have volume near M^3, but surface area small.

This result is one that I saw in a Colloquium back in Claremont. It is a interesting exercise to do the two dimensional case. The lines grow infinitely long, but one pauses at the ends to leave time for the next step.The area is near one, but the curve is no longer onto.

Most students in Calculus 2, see that the volume of $y = 1/x, 1 \leq x < \infty$ rotated about the x-axis has finite volume π but infinite surface area. A figure you can fill with paint, but you can't paint the surface. This solid is named Gabriel's horn and/or Torricelli's trumpet.

Picture sources

all pictures were done by the author using Scilab.