The Good, the Bad and the Ugly

This welcome is brought to you
by enough eigenvectors (the good),
complex eigenvalues (the bad),
and too few eigenvectors (the ugly)

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Eigenstuff

$$A \begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \lambda_1 e_1 & \lambda_2 e_2 \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$AP = PD \text{ or } A = PDP^{-1}$$

Let P be the matrix with columns eigenvectors e_1 and e_2. Here is the best way to think about eigenvalues and eigenvectors. We can write both $Ae_1 = \lambda_1 e_1$ and $Ae_2 = \lambda_2 e_2$ with the first matrix equation. Another way of the same result, is by multiplying by the diagonal matrix on the right. Multiplying by P inverse on the right, gives the similar bases form $A = PDP^{-1}$.

While this is 2×2 matrices, it generalizes to more variables. This is the good case, when there are enough independent eigenvectors.

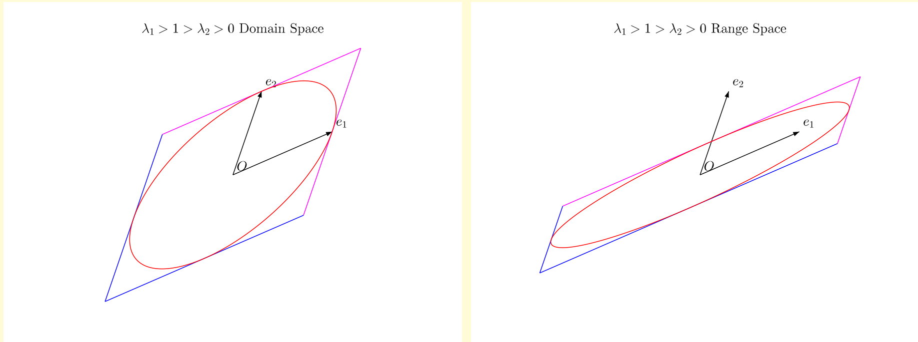
Commuting Diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{P^{-1}} & \mathbb{R}^2 \\ \downarrow A & & \downarrow D \\ \mathbb{R}^2 & \xleftarrow{P} & \mathbb{R}^2 \end{array}$$

One way of viewing diagonal matrices is by this diagram. It says we can, think of the map as a diagonal map with respect to the eigenvectors. Diagonal maps have easy to understand geormetries. Cases can have the eigenvalues can be both positive, both negative, and one of each.

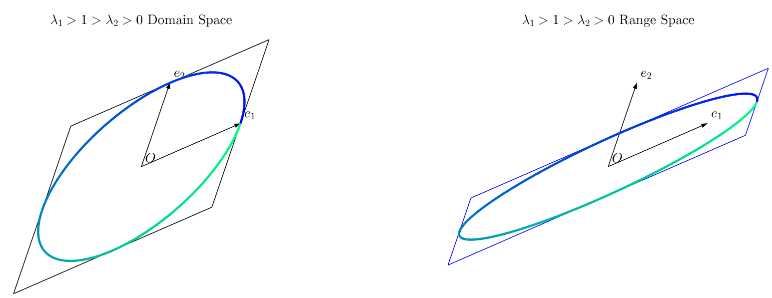
This commuting diagrams says these diagonalizable maps, look like diagonal operators in standard way.

Graphically

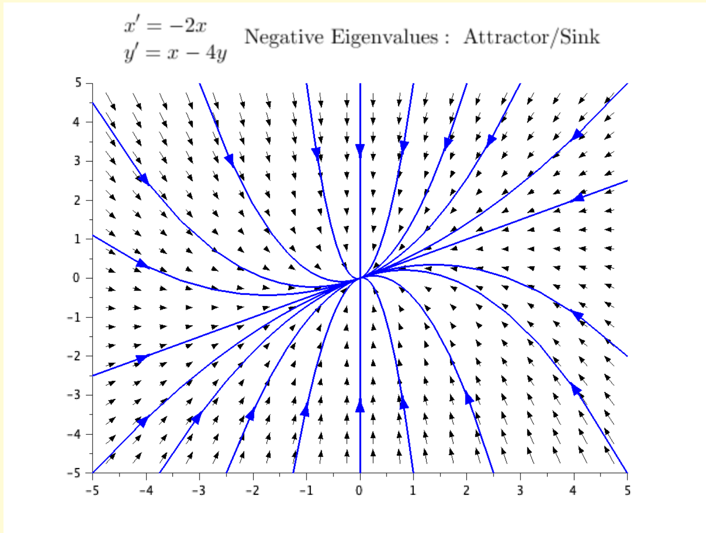


Here we draw figures in the domain, and the resulting images in the range. Ellispes go to ellispes. The rectangular frame is color coded, to show that λ_2 is positive. A negative λ_2 would swap the red and blue edges.

Here another picture with a just a color gradient to show the direction on the ellispe. Making this plot similar to latter pictures.



System $X'(t) = AX(t)$

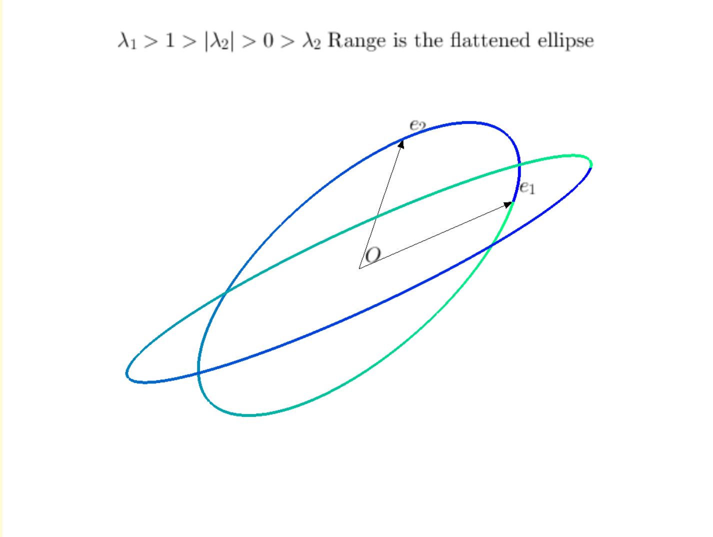


Often viewing the flow lines of the ODE system reveals the geometry. The matrix A has two negative eigenvalues. The dynamics is called an attacter or a sink, everything is moving towards $(0, 0)$. This shows both eigenvectors, and you can see the eigenvalue for the y-axis is more negative by the shape of the flows. Two positive eigenvalues would reverse the flow creating a source or a repeller.

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -4 \end{bmatrix}$$

has eigenvalues -2 and -4, and eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Opposite signs



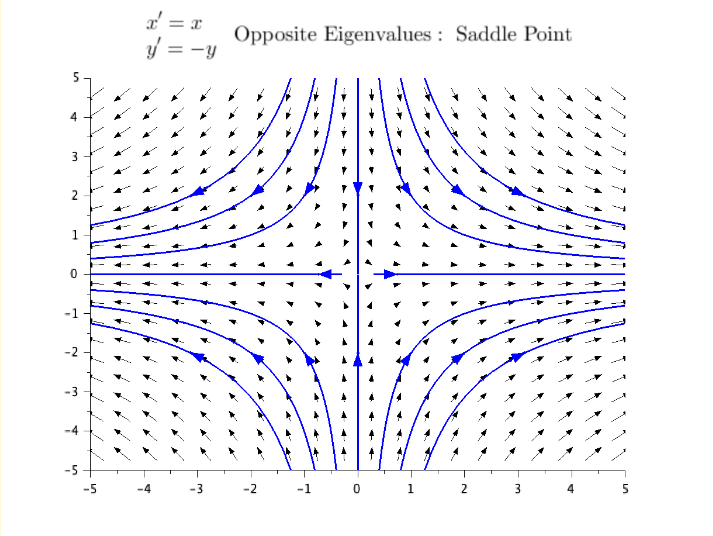
The more flatten ellispe is the range of the other ellipse. Note that the direction of the gradient is correctly showing the flip of the e_2 direction.

The map was construct by setting $\lambda_1 = 1.5, \lambda_2 = -0.4$ and picking eigenvectors

$$e_1 = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} \qquad e_2 = \begin{bmatrix} 1/4 \\ \sqrt{15}/4 \end{bmatrix} \qquad P = \begin{bmatrix} \sqrt{3}/2 & 1/4 \\ 1/2 & \sqrt{15}/4 \end{bmatrix}$$

$$A = P \begin{bmatrix} 1.5 & 0 \\ 0 & -0.4 \end{bmatrix} P^{-1} \approx \begin{bmatrix} 1.8328543 & -0.5765205 \\ 1.289139 & -0.7328543 \end{bmatrix}$$

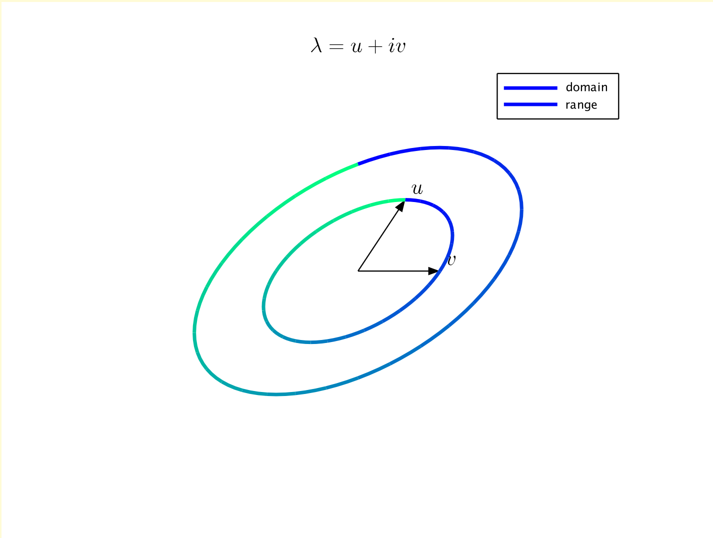
Saddle Points



Again viewing the flow lines of the ODE system reveals the geometry: a saddle point.

The vector $e_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has eigenvalue 1 and The vector $e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has eigenvalue -1.

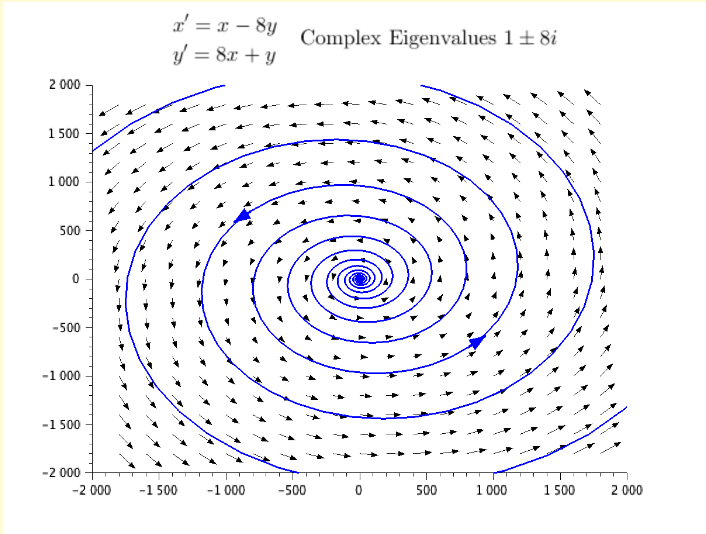
The Bad?



The map is a rotation followed by a scaling.

What was the matrix?

Complex eigenvalues

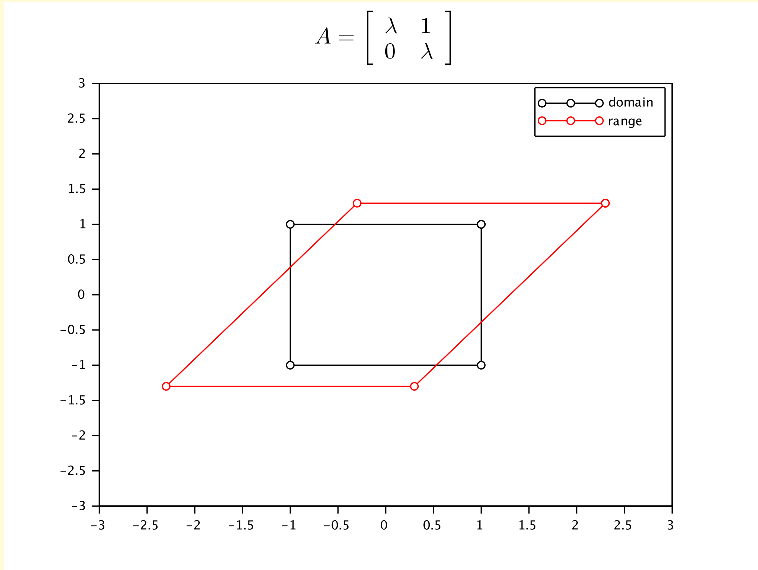


The map rotates counterclockwise and expands. Viewing the flow lines of the ODE systems reveals the spiral outward.

$$A = \begin{bmatrix} 1 & -8 \\ 8 & 1 \end{bmatrix}$$

has eigenvalues $1 \pm 8i$ with eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

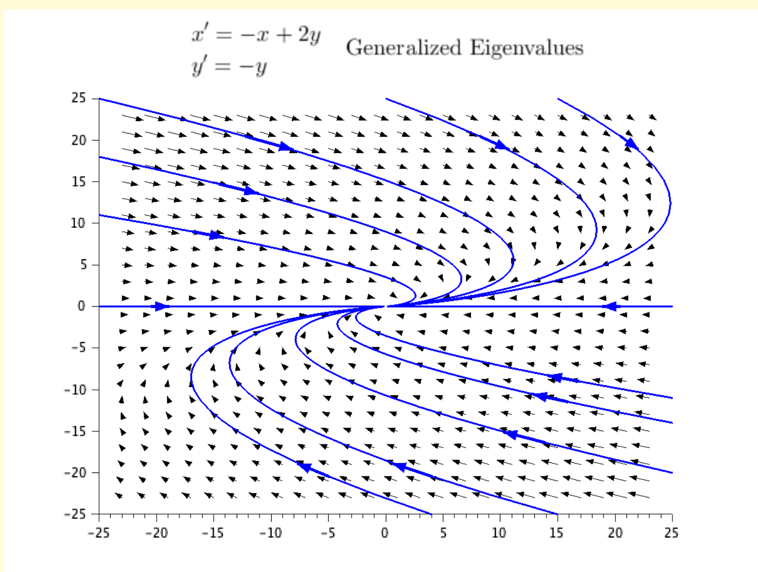
Generalized Eigenvectors



That is what I said.

new stuff

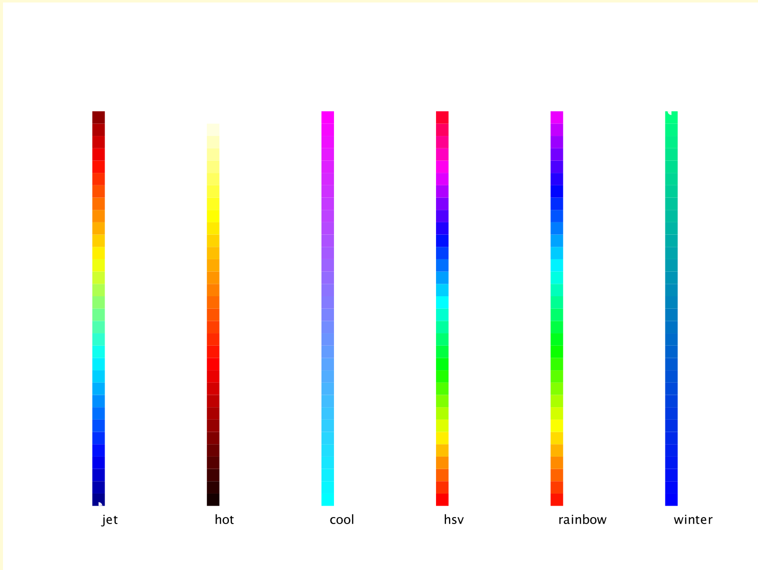
Differential systems



That is what I said.

new stuff

Colormap Gradients



There are a number of built in gradients. I used winter because of the simplicity of the gradient.

The *What is this class about handout* for Linear Algebra 2 `https://www.math.fsu.edu/~bellenot/class/s20/la2/goodbadugly.pdf`Eigenstuff: the Good, the Bad and the Ugly

Picture sources

pictures by the author using scilab.
Phase diagrams modeled on examples from `http://personal.psu.edu/sxt104/class/Math251/Notes-PhasePlane.pdf`