Math and Marriage

This welcome is brought to you
by Shapley and Gale,
the stable marriage algorithm,
and the wife picking problem.

Steven F. Bellenot

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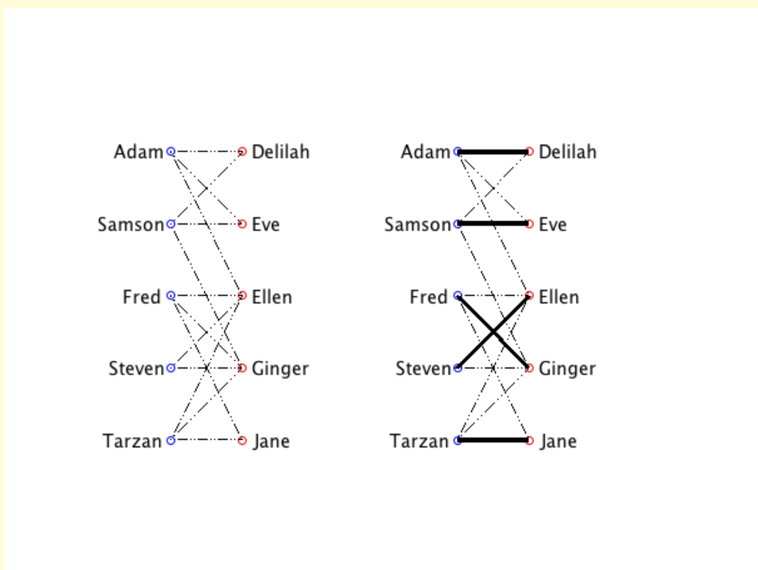
The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Assignment Problems and Matching



Unstable if Eve prefers Adam to Samson and Adam prefers Eve to Delilah.

There are two graphs. The one on the left is a bipartite graph showing the possible matches from the left side to the right side. In this a marriage problem, which has me, Steven with two possible pairing, my wife Ellen or Ginger Rogers. The graph on the right has matching, each person is matched with exactly one person on the other side. The assignment problem is to produce a best matching.

This matching could be unstable if Eve prefers Adam to Samson and Adam prefers Eve to Delilah. A stable matching is better than an unstable one.

Note the equal number of vertices in each partite set, which simplifies the discussion. Male vertices are blue, Female vertices are pink.

Stable Matchings

Men	Women
A: XYZ	X: BCA
B: YZX	Y: CAB
C: ZXY	Z: ABC

AX, BY, CZ	Men's first choices, Women's last
AY, BZ, CX	everyone's second choice
AZ, BX, CY	Women's first choices, Men's last

The top table lists the prefers of each person. For example A prefers the women in the order XYZ. He likes X the best and Z the least.

The bottom table gives three different stable matchings. The men don't want to change in the first, the women don't want to change in the last. The middle case takes a little more checking but it too is stable.

A stable matching doesn't mean everone is happy.

Its a love hexagon: A loves X, X loves B, B loves Y, Y loves C, C loves Z, Z loves A.

Gale Shapley Algorithm (1962)

repeat
  Men propose to their favorite among women that haven't rejected them
  Woman says "maybe" to her favorte proposer and rejects the rest
until there are no rejections (and everyone is promised)

Women "jilt" old favorites.
Picks the best stable matching for men, worst stable matching for women.
Nobel prize in 2012 (50 years later) for Shapley

Note that this algoritm (and others) won Shapley a share of the Nobel prize in economics 50 years later. My first publication was in 1974, so in 4 years maybe there is a Nobel prize for me. Gale has passed away before 2012. Nobel prizes are not awarded posthumously.

The algorithm has the men proosing to their favorite that hasn't rejected him. Women with multiple suitors, reject everyone but the current best. She may get a better offer in a later round. Eventually every woman gets a single offer.

This algorithm has been used to match hospital residencies to medical students. The algorithm favors the hospitals.

Note that applying the algorithm to ABC and XYZ, yields the men's first choice. However, if X goes rogue and rejects A's first offer, then the algorithm yields everyone's second choice. If she rejects C too, then the algorithm yields the women's first choice.

Wife Picking

W

Index (p 924): University Mathematics, R. C. James, 1963

This was my undergraduate calculus textbook. The index of the book lists a problem as the wife picking problem. It is also called the secretary problem. Lets look at the problem.

Given n balls, we need to pick p, to optimize our luck. If $j > p$, then the j is the largest and is picked is.

Page 661, Problem 6

6. A bag contains n balls, no two of which are the same size and the sizes of which are unknown. The balls are drawn from the bag one at a time. Suppose you wish to choose the largest ball, but must make the choice when the ball is drawn and before seeing the next ball. All that is known is the sizes of the balls that have been drawn. It can be shown that the best strategy is to pick a certain integer p, let the first p balls pass, and then pick the first ball that is larger than any of the first p balls. Show that the following are true.

(a) If $j > p$, then the probability the jth ball is the largest and is picked is equal to

$$\frac{1}{n}\frac{p}{j-1}.$$

(b) The probability the largest ball is picked is equal to

$$\frac{p}{n}\left[\sum_{p+1}^{n}\frac{1}{j-1}\right].$$

There is no mention of spouses or partners in this problem. How did it get found? It didn't take long, within a few weeks.

To see part a, note the probability of the jth ball being the largest is $1/n$ and it will picked if the largest of the $j-1$ is one of the first p balls, which has probability $p/(j-1)$.

Part b follows being the sum of these probabilities.

Integral Estimate

Since $1/t$ is decreasing, Left Sum >Integral >Right Sum. Use $x = p/n$ and $\Delta t = 1/n$.

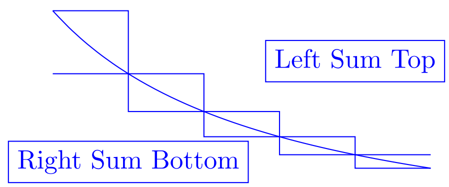
$$x \int_x^1 \frac{dt}{t} = x \ln t|_x^1 = x \ln 1 - x \ln x = -x \ln x$$

$$\frac{p}{n} \sum_{j=p+1}^{n} \frac{1}{(j-1)/n} \frac{1}{n} > -x \ln x > \frac{p}{n} \sum_{j=p+1} \frac{1}{j/n} \frac{1}{n}$$

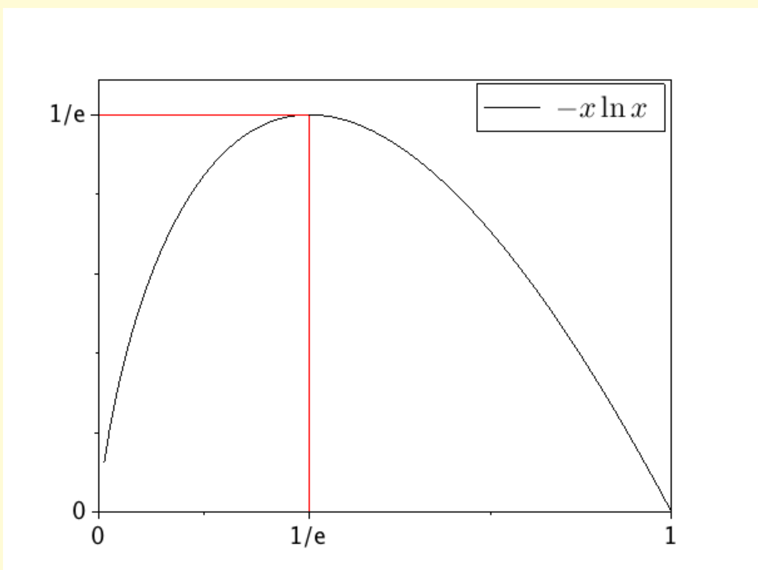
$$\frac{p}{n} \sum_{j=p+1}^{n} \frac{1}{(j-1)} > -x \ln x > \frac{p}{n} \sum_{j=p+1} \frac{1}{j}$$

Here we show $f(x) = -x \ln x$ is a good estimate of our answer in b. The function $1/t$ is decreasing so its integral is trapped between the Left Sum and Right Sum.

Here is a visual aid to see the relationships between the integral, the left sum and the right sum.



The $1/e \approx 37\%$ Rule



The graph of $f(x) = -x \ln x$ has a maximum or $1/e \approx 0.37$ at $x = 1/e = p/n$.

So the best number for $p = 0.37n$. No matter what n is, watch a little over $1/3$ go by. And then pick the next bigger ball.

Over $2/3$ of the time, the largest ball is not selected. Could this be why many marriages end?

This is almost soap opera worthly. Of course it is not possible to linearly order people. And people tend to not marry their first date. Most date more than one person, before finding their one true love.