Honors Day 2012
One sixth of a squared π

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Euler and the Basel Problem

1735 Euler "solved" the famous Basel Problem and announced to world

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

A rigorous proof had to wait for several years. Euler found a way to suggest this answer. His numerical 19 digit approximation to the sum matched 18 digits of the value $\pi^2/6$.

Don't Try This at Home

Professional Driver on a Closed Course

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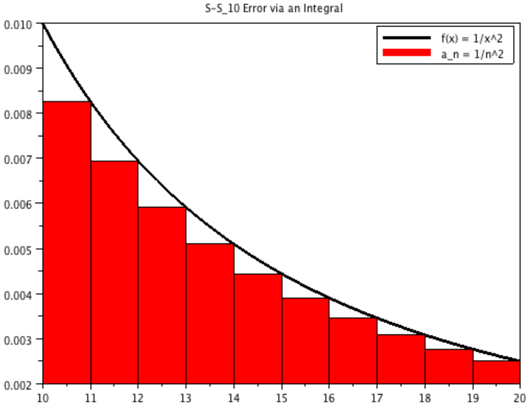
Error Estimate $S - S_{10}$



How many terms?

Estimate the error using the integral test.

$$S - S_N = \sum_{n=N+1}^{\infty} \frac{1}{n^2} \qquad \leq \int_N^{\infty} \frac{dx}{x^2} \leq \left. \frac{-1}{x} \right|_N^{\infty} = \frac{1}{N}$$

To get 18 digits you need $N = 10^{17}$.

C code

```
int limit = 1000000000;/* 10^9 = 2^30*/

for ( i=1; i <= limit; i++ ) {
        sum1 += 1.0/(i*i);
}

for ( i=limit; i > 0; i-- ) {
        sum2 += 1.0/(i*i);
}

printf("sum1 is %1.17lf\n", sum1);
printf("sum2 is %1.17lf\n", sum2);
printf("pi^2/6 =%1.17lf\n", M_PI*M_PI/6);
```

C code output

sum1 is 1.644934057 83457503
sum2 is 1.644934065 84822632
answ is 1.644934066 84822641
run time 0m23.184s
For 18 digits 7.35 years.
M_ PI = 3.14159265358979323846264338327950288

Outline

- Approximating π to hundreds of digits.
- Approximating the sum to 6 digits
- Approximating the sum to 18 digits
- Euler's first proof?
- A modern proof.

Computing π

The Taylor series for $\arctan x$ about $a = 0$ converges for $|x| \leq 1$ (Known to Gregory (1638-1675) and Leibniz (1646 - 1716))

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

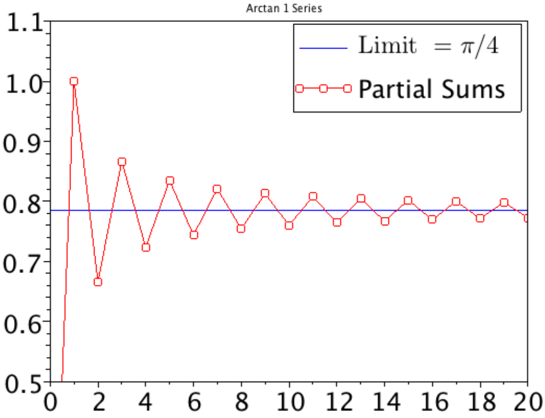
$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1^3}{3} + \frac{1^5}{5} - \cdots$$

The error in this approximation is roughly the first term omitted because it is an alternating sum. To get 10 digits correct,

$$\frac{1}{2n-1} < 10^{-10} \text{ or } 5000000000 < n$$

one must sum five billion terms.
(If $x = 1/2$, we would only need 15 terms)

Error estimate for Alternating Series



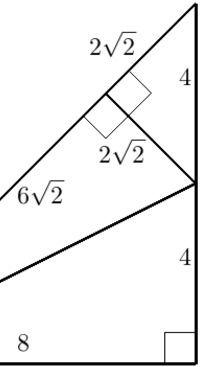
Error is less than the first term omitted.

A geometric proof

$$\alpha = \arctan\frac{1}{2}$$

$$\beta = \arctan\frac{1}{3}$$

$$\frac{\pi}{4} = \alpha + \beta$$



A algebraic proof

The sum formula for tangent

$$\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$$

Let $x = \tan \beta$ and since $\alpha + \beta = \pi/4$ and $\tan \alpha = 1/2$

$$\begin{aligned} 1 &= (1/2 + x)/(1 - (1/2)x) \\ 1 - x/2 &= 1/2 + x \\ 1/2 &= 3x/2 \\ x &= 1/3 \end{aligned}$$

arctan 1/2 and arctan 1/3

0.50000000	0.33333333
0.04166667	0.01234568
0.00625000	0.00082305
0.00111607	0.00006532
0.00021701	0.00000565
0.00004439	0.00000051
0.00000939	0.00000005
0.00000203	0.00000000
0.00000045	0.00000000
0.46364769	0.32175055
3.14159298	

In 1706, John Machin discovered the formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

and used it to compute 100 digits of π

Tangent formulas

$$\tan(\alpha \pm \beta) = (\tan\alpha \pm \tan\beta)/(1 \mp \tan\alpha \tan\beta)$$

$$\begin{aligned}
\theta &= \arctan(1/5) \\
\tan 2\theta &= (2/5)/(1-(1/5)^2) = 10/24 = 5/12 \\
\tan 4\theta &= (10/12)/(1-(5/12)^2) = 120/119 \\
1 &= (120/119 - x)/(1 + 120x/119) \\
1 &= (120 - 119x)/(119 + 120x) \\
(119 + 120x) &= (120 - 119x) \\
239x &= 1 \\
x &= 1/239
\end{aligned}$$

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239)$$

arctan 1/5 and arctan 1/239

0.200000000000000	0.004184100418410
0.002666666666667	0.000000024416592
0.000064000000000	0.000000000000256
0.000001828571429	0.000000000000000
0.000000056888889	0.000000000000000
0.000000001861818	0.000000000000000
0.000000000063015	0.000000000000000
0.000000000002185	0.000000000000000
0.000000000000077	0.000000000000000
0.197395559849883	0.004184076002075

3.141592653589 836

Best formula for π til the 1960's

1000 digits of π

3.14159265358979323846264338327950288419716939937510
58209749445923078164062862089986280348253421170679
82148086513282306647093844609550582231725359408128
48111745028410270193852110555964462294895493038196
44288109756659334461284756482337867831652712019091
45648566923460348610454326648213393607260249141273
72458700660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415116094
33057270365759591953092186117381932611793105118548
07446237996274956735188575272489122793818301194912
98336733624406566430860213949463952247371907021798
60943702770539217176293176752384674818467669405132
00056812714526356082778577134275778960917363717872
14684409012249534301465495853710507922796892589235
42019956112129021960864034418159813629774771309960
51870721134999999837297804995105973173281609631859
50244594553469083026425223082533446850352619311881

3.1415926535897932384626433832795028841971693993751
05820974944592307816406286208998628034825342117067
98214808651328230664709384460955058223172535940812
84811174502841027019385211055596446229489549303819
64428810975665933446128475648233786783165271201909
14564856692346034861045432664821339360726024914127
37245870066063155881748815209209628292540917153643
67892590360011330530548820466521384146951941511609
43305727036575959195309218611738193261179310511854
80744623799627495673518857527248912279381830119491
29833673362440656643086021394946395224737190702179
86094370277053921717629317675238467481846766940513
20005681271452635608277857713427577896091736371787
21468440901224953430146549585371050792279689258923
54201995611212902196086403441815981362977477130996
05187072113499999987297804995105973173281609631859
50244594553469083026425223082533446850352619311881

1000 digits of π

3.1415926535897932384626433832795028841971693993751
05820974944592307816406286208998628034825342117067
98214808651328230664709384460955058223172535940812
84811174502841027019385211055596446229489549303819
64428810975665933446128475648233786783165271201909
14564856692346034861045432664821339360726024914127
37245870066063155881748815209209628292540917153643
67892590360011330530548820466521384146951941511609
43305727036575959195309218611738193261179310511854
80744623799627495673518857527248912279381830119491
29833673362440656643086021394946395224737190702179
86094370277053921717629317675238467481846766940513
20005681271452635608277857713427577896091736371787
21468440901224953430146549585371050792279689258923
54201995611212902196086403441815981362977477130996
05187072113499999987297804995105973173281609631859
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98214808651328230664709384460955058223172535940812
84811174502841027019385211055596446229489549303819
64428810975665933446128475648233786783165271201909
14564856692346034861045432664821339360726024914127
37245870066063155881748815209209628292540917153643
67892590360011330530548820466521384146951941511609
43305727036575959195309218611738193261179310511854
80744623799627495673518857527248912279381830119491
29833673362440656643086021394946395224737190702179
86094370277053921717629317675238467481846766940513
20005681271452635608277857713427577896091736371787
21468440901224953430146549585371050792279689258923
54201995611212902196086403441815981362977477130996
05187072113499999987297804995105973173281609631859
50244594553469083026425223082533446850352619311881

Euler's first estimate of the sum

Some power series tricks. (Radius of convergence is 1.)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$\frac{-\ln(1-x)}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}$$

$$\int_0^x -\frac{\ln(1-t)}{t}\,dt = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

$$\mathrm{Li}_2(x) = \int_0^x -\frac{\ln(1-t)}{t}\,dt$$

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Euler's first estimate of the sum, cont.

$$\begin{aligned}\mathrm{Li}_2(1) &= \int_0^1 -\frac{\ln(1-t)}{t}\,dt \\ &= \int_0^x -\frac{\ln(1-t)}{t}\,dt + \int_x^1 -\frac{\ln(1-t)}{t}\,dt \\ &= \mathrm{Li}_2(x) + \int_x^1 -\frac{\ln(1-t)}{t}\,dt\end{aligned}$$

Substitute $u = 1 - t$ on the remaining integral.

$$\begin{aligned}\int_x^1 -\frac{\ln(1-t)}{t}\,dt &= \int_{1-x}^0 -\frac{\ln(u)}{1-u}\,(-du) \\ &= \int_0^{1-x} -\frac{\ln(t)}{1-t}\,dt\end{aligned}$$

Euler's first estimate of the sum, cont.

Integrate by parts $u = \ln(t), dv = -dt/(1-t), v = \ln(1-t)$

$$\int_0^{1-x} -\frac{\ln(t)}{1-t}\,dt = \ln(t)\ln(1-t)\big|_0^{1-x} - \int_0^{1-x} \frac{\ln(1-t)}{t}\,dt$$
$$= \ln(1-x)\ln(x) + \int_0^{1-x} -\frac{\ln(1-t)}{t}\,dt$$
$$= \ln(1-x)\ln(x) + \mathrm{Li}_2(1-x)$$

The lower limit is improper but has value 0

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+} \frac{x^2}{-x} = \lim_{x\to 0^+} -x = 0$$

$$\mathrm{Li}_2(1) = \mathrm{Li}_2(x) + \ln(x)\ln(1-x) + \mathrm{Li}_2(1-x)$$

Euler's first estimate of the sum, cont.

Using $x = 1/2$

$$\mathrm{Li}_2(1) = \ln^2(1/2) + 2\mathrm{Li}_2(1/2)$$

Error estimate II: Geometric

Suppose $f(x)$ is a positive decreasing function and $x = 1/M$ for some integer with $M \geq 2$, then

$$\sum_{n=N+1}^{\infty} f(n)x^n \leq f(N+1)x^{N+1} \sum_{n=0}^{\infty} x^n \leq f(N+1)x^{N+1}/(1-x)$$

The error in using S_N to estimate S is less than

$$\frac{f(N+1)}{M^{N+1}} \frac{M}{M-1} \leq 2\frac{f(N+1)}{M^{N+1}}$$

less than twice the first term omitted.

The first estimate

i	$\ln(1/2)$	$\mathrm{Li}_2(1/2)$
1	0.500000 00	0.500000 00
2	0.125000 00	0.062500 00
3	0.041666 67	0.013888 89
4	0.015625 00	0.003906 25
5	0.006250 00	0.001250 00
6	0.002604 17	0.000434 03
7	0.001116 07	0.000159 44
8	0.000488 28	0.000061 04
9	0.000217 01	0.000024 11
10	0.000097 66	0.000009 77
11	0.000044 39	0.000004 04
12	0.000020 35	0.000001 70
13	0.000009 39	0.000000 72
14	0.000004 36	0.000000 31
15	0.000002 03	0.000000 14

The second estimate

+1.549767731166540690
+0.10000000000000000
-0.00500000000000000
+0.00016666666666666
-0.00000333333333333
+0.00000002380952381
-0.00000000033333333
+0.00000000000757575
-0.00000000000025311
+0.00000000000001166
-0.00000000000000071
=1.644934066848226430

=1.644934066848226436

Euler Maclaurin Summation

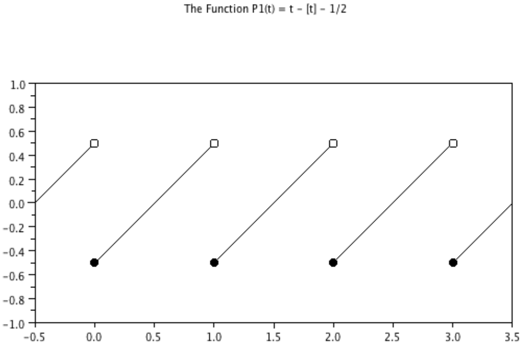
The first term is $\sum_{n=1}^{9} 1/n^2$, the remaining terms are from the Euler-Maclaurin Summation formula with $f(x) = 1/(10+x)^2$

$$f(0)+f(1)+\dots f(n) = \int_0^n f(t)\,dt + (f(n)+f(0))/2 + \int_0^n P_1(t)f'(t)\,dt$$

$P_1(t) = t - \lfloor t \rfloor - 1/2$
Let $P_2(t)$ be $P_2'(t) = P_1(t)$ and so that $\int_0^1 P_2(t)\,dt = 0$ and eventually $P_2(t)$ is periodic with period one.

$$\int_0^n P_1(t)f'(t)\,dt = P_2(t)f'(t)\Big|_0^n - \int_0^n P_2(t)f''(t)\,dt$$

$$\int_0^n P_1(t)f'(t)\,dt = P_2(0)(f'(n) - f'(0)) - \int_0^n P_2(t)f''(t)\,dt$$


The Function P1(t) = t - [t] - 1/2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
−0.8
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

The proof?

If the polynomial $p(x) = 1 + a_1 x + ... a_n x^n$ has complex roots at $z_1, z_2, \ldots z_n$ then

$$p(x) = (1 - x/z_1)(1 - x/z_2) \ldots (1 - x/z_n)$$

and $a_1 = -\sum_{i=1}^{n} 1/z_i$
Now $\sin x/x$ has zero's at $n\pi$, $n = \pm 1, \pm 2 \ldots$ and $sin 0/0 = 1$ use

$$(1 - x/(n\pi))(1 + x/(n\pi)) = 1 - x^2/(n^2\pi^2)$$

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} (1 - \frac{x^2}{n^2\pi^2})$$

Problems: 1. This “proof” also implies $\exp(x)\sin(x)/x$ has the same product representation. 2. Are these all the complex zeros of $\sin(x)$? 3. It applies a result for polynomials to an infinite series. 4 Conditionally convergent product

The proof? cont

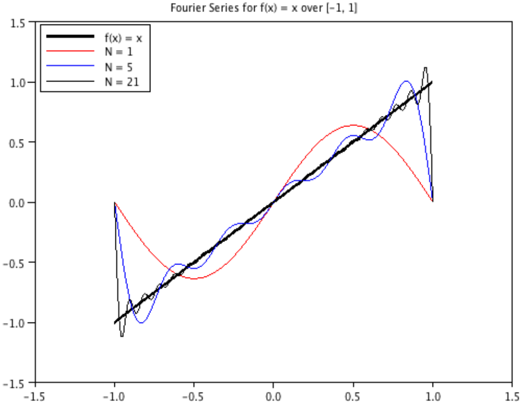
$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2})$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} \cdots$$

$$\prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2}) = 1 - x^2\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2} \cdots$$

$$\frac{1}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}$$

The Fourier series of $f(x) = x, x \in [-1, 1]$



Computing the Fourier Series

$f(x)$ is an odd function so all the cos coefficents are zero.

$$
\begin{aligned}
b_n &= \int_{-1}^{1} x\sin(n\pi x)\,dx \\
&= 2\int_{0}^{1} x\sin(n\pi x)\,dx \\
&= 2\left(-\frac{x\cos(n\pi x)}{n\pi}\bigg|_0^1 + \int_0^1 \frac{\cos(n\pi x)}{n\pi}\,dx\right) \\
&= 2\left(-\frac{\cos(n\pi)}{n\pi} + \frac{\sin(n\pi x)}{n^2\pi^2}\bigg|_0^1\right) \\
&= \frac{2(-1)^{n+1}}{n\pi}
\end{aligned}
$$

$$
f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi}\sin(n\pi x)
$$

Compute The Norm $\|f\|^2$

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

$$\int_{-1}^{1} x^2\,dx = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \int_{-1}^{1} \sin^2(n\pi x)\,dx$$

$$\left.\frac{x^3}{3}\right|_{-1}^{1} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3}\frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Jokes?

- Politicians. The Indiana House voted to make π a different value:
  It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square of one side.
- $A = \pi r^2$ No, cake are square, pie are round.
- π day is March 14 is 3/14 in America, But Mar 14 is 14/3 in Europe. Will 31/4 be π day in France? Maybe 22/7 or July 22 will be a rational approximation π day.