Honors Day 2013
Transcending the Irrationality

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λ, e, π are transcendental

Liouville, λ, 1844/51

Hermite, e, 1873

Lindemann, π, 1882

Cantor (1874) Non-constructively:
the transcendental numbers are uncountable.

Rational Numbers

A number α is *rational* if for some integers p and q

$$\alpha = \frac{p}{q}$$

otherwise α is *irrational*.
Note α is root of the degree 1 integer-coefficient polynomial:

$$f(x) = qx - p$$

Algebraic Numbers

A number α is *algebraic* if it is a root of a polynomial $f(x)$ with integer coefficients, otherwise α is *transcendental*.
The *degree* of a an algebraic number α is is the smallest degree of an all integer coefficient polynomial $f(x)$ so that $f(\alpha) = 0$.
If the degree is one, then α is rational.
Example: $\sqrt{2}$ is a root of $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$ so $\sqrt{2}$ is algebraic of degree no more than 2. The degree will be exactly two; once we show $\sqrt{2}$ is irrational.

$\sqrt{2}$ is irrational

There is an $\varepsilon > 0$ and a $C > 0$ so that for integers p and q,

$$\left|\sqrt{2} - \frac{p}{q}\right| < \varepsilon \implies \left|\sqrt{2} - \frac{p}{q}\right| > \frac{C}{q^2}$$

The statement above is stronger than saying $\sqrt{2}$ is irrational. I called it the “it ain’t me babe” inequality.

The $\varepsilon > 0$ condition is a dodge; an unnecessary condition: it is a straightforward exercise to show there is a $C' > 0$

$$\left|\sqrt{2} - \frac{p}{q}\right| \geq \varepsilon \implies \left|\sqrt{2} - \frac{p}{q}\right| > \frac{C'}{q^2}$$

Indeed, any positive $C' < \varepsilon$ works.

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Indeed, any positive $C' < \varepsilon$ works.

$\sqrt{2}$ is irrational

For integers p and q with $|\sqrt{2} - p/q| < \varepsilon = 0.08$, then for $C = 1/3$,

$$|\sqrt{2} - \frac{p}{q}| \geq \frac{C}{q^2}$$

$$\left|\sqrt{2} - \frac{p}{q}\right| = \left|(\sqrt{2} - \frac{p}{q})\frac{\sqrt{2} + \frac{p}{q}}{\sqrt{2} + \frac{p}{q}}\right| \geq \frac{|2q^2 - p^2|}{3q^2} \geq \frac{1}{3q^2}$$

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The Key Idea

The number $\sqrt{2}$ is *far away* from any rational number.

Perhaps the same is true for irrational algebraic numbers.

There are numbers that are not so far away and hence would be transcendental.

Liouville: λ is transcendental

$$\lambda = \sum_{n=1}^{\infty} 10^{-n!}$$

$\lambda =$ sum of

0.1

0.01

0.000001

0.000000000000000000000001

0.0000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000

. . .

λ is zero in decimal places from $n! + 1$ to $(n+1)! - 1$ which is $n!n - 2$ zeros.

$$\lambda = 0.110001\,\underbrace{0\ldots0}_{16\ 0's}1\underbrace{0\ldots0}_{94\ 0's}1\underbrace{0\ldots0}_{598\ 0's}1\,\underbrace{0\ldots0}_{4318\ 0's}1\underbrace{0\ldots0}_{34278}1\underbrace{0\ldots0}_{322558}10\ldots$$

Estimate

Let $q_k = 10^{k!}$, then there are integers p_k and q_k

$$\frac{p_k}{q_k} = \sum_{n=1}^{k} 10^{-n!}$$

Note

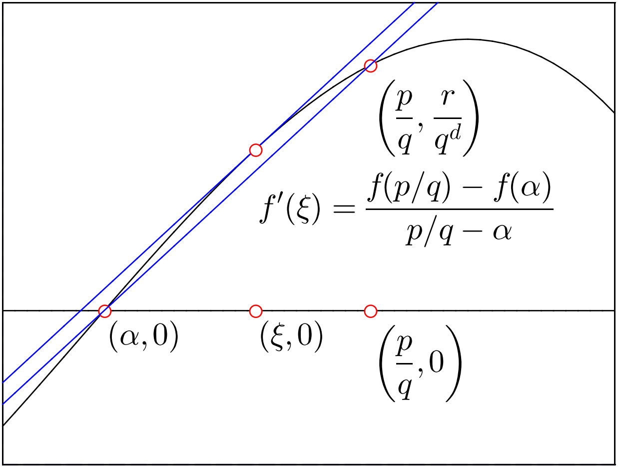
$$q_{k+1} = 10^{(k+1)!} = (10^{k!})^{k+1}$$

So there are integers p_k and $q_k = 10^{k!}$ with

$$|\lambda - \frac{p_k}{q_k}| < \frac{2}{q_k^{k+1}}$$

Insight: irrational algebraic numbers cannot be so well approximated by rationals.

Mean Value Theorem



α a root of $f(x)$, a polynomial of degree d

$f(p/q) = r/q^d$

$$f(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots a_1 x + a_0$$

$$f\left(\frac{p}{q}\right) = a_d\left(\frac{p}{q}\right)^d + a_{d-1}\left(\frac{p}{q}\right)^{d-1} + \cdots a_1\left(\frac{p}{q}\right) + a_0$$

$$f\left(\frac{p}{q}\right) = \frac{a_d p^d + a_{d-1}p^{d-1}q + \cdots a_1 pq^{d-1} + a_0 q^d}{q^d}$$

So $f(p/q) = r/q^d$ where

$$r = a_d p^d + a_{d-1}p^{d-1}q + \cdots a_1 pq^{d-1} + a_0 q^d$$

If p/q is not a root of f, then $|f(p/q)| > 1/q^d$

Side Note About Multiple Roots

If $f(x)$ is a polynomial where α is a multiple root, then

$$f(x) = (x - \alpha)^2 g(x)$$

and so $f'(x)$ also has α as root. The converse is also true. So each algebraic number α is a single root of the polynomial $f(x)$ of of minimal degreee d so that $f(\alpha) = 0$. (If it is not, replace f by the smaller degree f', a contradiction)

Liouville's Estimate

Theorem. If the algebraic number α has degree d, then there is an $\varepsilon > 0$ and $C > 0$, so for all integers p and q we have

$$\left|\alpha - \frac{p}{q}\right| < \varepsilon \implies \left|\alpha - \frac{p}{q}\right| > \frac{C}{q^d}$$

Proof: Let $\varepsilon > 0$, so that $f(x) \neq 0$ for $0 < |x - \alpha| < \varepsilon$. There is a C so that $|f'(x)| < C$, for $|\alpha - x| < \varepsilon$. Apply the MVT:

$$\left|\frac{f(\alpha) - f\left(\frac{p}{q}\right)}{\alpha - \frac{p}{q}}\right| = |f'(\xi)| < C$$

$$\frac{1}{q^d} \leq |f\left(\frac{p}{q}\right)| < C\left|\alpha - \frac{p}{q}\right|$$

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The contradiction

$$\frac{1}{q_k^d} \leq C\left|\alpha - \frac{p_k}{q_k}\right| < C\frac{2}{q_k^{k+1}}$$

But this is impossible since it implies

$$1 < 2Cq_k^{d-k-1}$$

but the RHS goes to zero as as $k \to \infty$

There are transcendental numbers

Since λ is not algebraic, transcendental numbers exist. However λ was created. How about an already existing number?

Your assignment

Exercise: Show $e = \sum_{i=0}^{\infty} 1/i!$ is irrational.
Who proved (and when) if α algebraic and irrational $\varepsilon > 0$, then there is a $C = C(\alpha, \varepsilon)$ so for all intergers p and q,

$$\left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2+\varepsilon}}$$

The contradiction:

$$\text{LHS} = \text{RHS}$$

We will show that the RHS is a non-zero integer and that

$$\lim_{p \to \infty} \text{LHS} = 0$$

using the assumption that for some integer coefficient polynomial

$$a_d e^d + a_{d-1} e^{d-1} + \cdots + a_1 e^1 + a_0 e^0 = 0$$

Product Rule

The n-derivative of $f = gh$ is
$f^{(1)} = g^{(1)}h^{(0)} + g^{(0)}h^{(1)}$
$f^{(2)} = g^{(2)}h^{(0)} + 2g^{(1)}h^{(1)} + g^{(0)}h^{(2)}$
$f^{(3)} = g^{(3)}h^{(0)} + 3g^{(2)}h^{(1)} + 3g^{(1)}h^{(2)} + g^{(0)}h^{(3)}$
...

$$f^{(n)} = \sum_{i=0}^{n} \binom{n}{i} g^{(n-i)}h^{(i)}$$

e is transcendental RHS

Lemma

if $h(x)$ is a polynomial with integer coefficients and $f(x) = (x - i)^p h(x)/(p-1)!$, then the j-th derivative at i, $f^{(j)}(i)$ is divisible by p.

The j-th derivative at i of $g(x) = (x - i)^p/(p-1)!$ is zero at i if $j < p$ or $j > p$ and is p when $j = p$. Each term in j-th derivative of f has one these terms as a factor.

e is transcendental RHS

Lemma

if $f(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p/(p-1)!$ *and* $p > m$ *is a prime then the* j*-th derivative* $f^{(j)}(0)$ *is not divisible by* p *for* $j \geq p-1$

The j-th derivative of $g(x) = x^{p-1}/(p-1)!$ is zero at 0 if $j < p-1$ or $j \geq p$ and is 1 when $j = p-1$. Hence each term in j-th derivative of f is either zero, or for large j, not divisible by p.

A basic limit

$$\lim_{p\to\infty} \frac{A^p}{(p-1)!} = 0$$

Eventually the terms decrease exponentially once $A/p < 1/2$.

$$\lim_{p\to\infty} \frac{m^{(m+1)p-1}}{(p-1)!} = 0$$

Let $A = m^{m+1}$

e is transcendental LHS

if $f(x) = x^{p-1}(x-1)^p(x-2)^p \cdots (x-m)^p/(p-1)!$ then on the interval $[0, m]$

$$|f(x)| < \frac{m^{mp-1}}{(p-1)!} \le \frac{A^p}{(p-1)!}$$

Let $A = m^m$.
Therefore

$$\max_{x \in [0,m]} |f(x)| \to 0 \text{ as } p \to \infty$$

e is transcendental

Let

$$F(x) = \sum_{n=0}^{\infty} f^{(n)}(x) \text{ where } f(x) = \frac{x^{p-1}(x-1)^p \cdots (x-m)^p}{(p-1)!}$$

Note the sum is finite as the derivatives $> mp + p - 1$ are zero.
Note

$$\frac{d}{dx}(e^{-x}F(x)) = e^{-x}[F'(x) - F(x)] = -e^{-x}f(x)$$

Therefore

$$a_j e^j \int_0^j e^{-x} f(x)\, dx = a_j e^j (-e^{-x}F(x))|_0^j = a_j e^j (F(0) - e^{-j}F(j))$$

Summing over j

$$\sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)\, dx = F(0) \sum_{j=0}^{m} a_j e^j - \sum_{j=0}^{m} a_j \sum_{i=0}^{\infty} f^{(i)}(j)$$

e is transcendental RHS

$$\sum_{j=0}^{m} a_j \sum_{i=0}^{\infty} f^{(i)}(j)$$

is an integer not divisible by p.

e is transcendental LHS

$$\left| \sum_{j=0}^{m} a_j e^j \int_0^j e^{-x} f(x)\, dx \right| < A \int_0^m |f(x)|\, dx \to 0$$

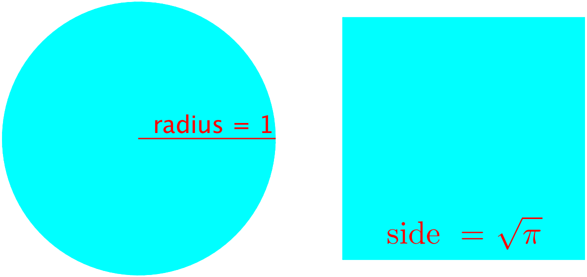
π is transcendental

Theorem

If α algebraic, then e^{α} is transcendental.

Assuming the theorem: if π algebraic, then so is $i\pi$. But this implies $e^{i\pi} = -1$ is transcendental.

Squaring the Circle



both figures have an area of π

Let's solve a 2000 year question

Corollary

One cannot square the circle.

Otherwise, one could construct $\sqrt{\pi}$ with a straight edge and compass which would require both $\sqrt{\pi}$ and π to be algebraic.

Jokes?

- $A = \pi r^2$ No pie's are round, cakes are square.
- π vs e, Which is the better transcendental number?
  Well π runs circles around e.
- The τ Manifesto. It should be $\tau = C/r$ and not $\pi = C/D$.

$$\tau = 2\pi,\ C = \tau r,\ \text{right angle = quarter of circle } = \tau/4$$

Euler's formula $e^{\tau i} = 1$

- To be politically correct:
  Transcendental numbers are polynomially challenged.

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