Honors Day 2014
Be a Part of the Story

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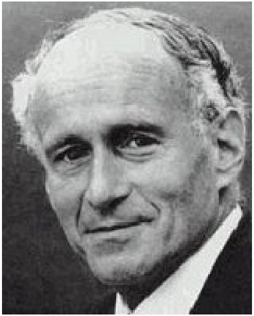
Famous uses of Integration by Parts



Laurent Schwartz (1915-2002), Fields Medal 1950

Peter Lax (1926-), National Medal of Science 1986

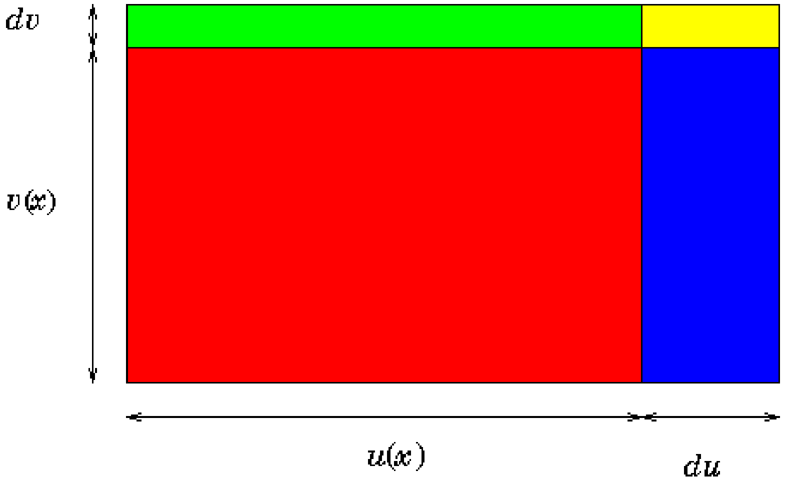
Gottfried Wilhelm von Leibniz, (1646-1716)

Three L's

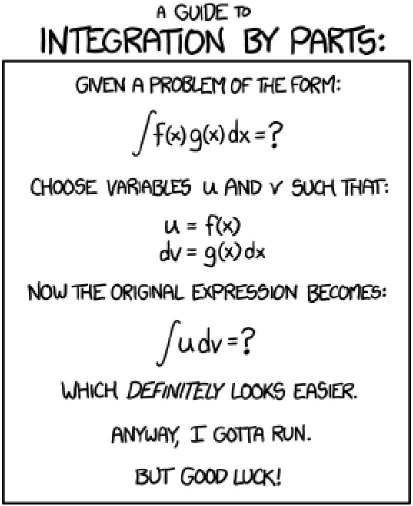
You are Really Famous when you have a Cookie

Lebnitz Product Rule

The derivative of uv is $u'v + uv'$ so $d(uv) = vdu + udv$.



xkcd

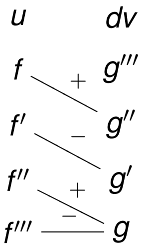

A GUIDE TO
INTEGRATION BY PARTS:
GIVEN A PROBLEM OF THE FORM:
∫f(x)g(x)dx = ?
CHOOSE VARIABLES u AND v SUCH THAT:
u = f(x)
dv = g(x)dx
NOW THE ORIGINAL EXPRESSION BECOMES:
∫u dv = ?
WHICH DEFINITELY LOOKS EASIER.
ANYWAY, I GOTTA RUN.
BUT GOOD LUCK!

If it is Nice, Do it Twice

$$\int fg' = fg - \int f'g$$

$$\int fg'' = fg' - f'g + \int f''g$$

$$\int fg''' = fg'' - f'g' + f''g - \int f'''g$$



If $f = x^2$ and $g''' = \cos nx$, we have:

$$\int x^2 \cos(nx) = x^2 \sin(nx)/n + 2x \cos(nx)/n^2 - 2\sin(nx)/n^3 - 0$$

The Table

u	dv
x^2	$\cos nx$
$2x$	$\sin nx/n$
2	$-\cos nx/n^2$
0	$-\sin nx/n^3$

u	sign	dv
x^2		$\cos nx$
$2x$	$+$	$\sin nx/n$
2	$-$	$-\cos nx/n^2$
0	$+$ / $-$	$-\sin nx/n^3$

$$\int x^2\cos(nx) = x^2\sin(nx)/n + 2x\cos(nx)/n^2 - 2\sin(nx)/n^3 - 0$$

Another Table

u	dv
e^{2x}	$\sin(3x)$
$2e^{2x}$	$-\cos(3x)/3$
$4e^{2x}$	$-\sin(3x)/9$

u		dv
e^{2x}	$+$	$\sin(3x)$
$2e^{2x}$	$-$	$-\cos(3x)/3$
$4e^{2x}$	$+$	$-sin(3x)/9$

$$\int e^{2x}\sin(3x) = -e^{2x}\cos(3x)/3 + 2e^{2x}\sin(3x)/9 - 4/9\int e^{2x}\sin(3x)$$

Gamma Function

$u = x^{t-1}$, $dv = e^{-x}$, $v = -e^{-x}$

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$$
$$\Gamma(t) = (t-1) \int_0^\infty x^{t-2} e^{-x}\, dx$$
$$\Gamma(t) = (t-1)\Gamma(t-1)$$
$$\Gamma(n) = (n-1)!$$

Inverse Functions

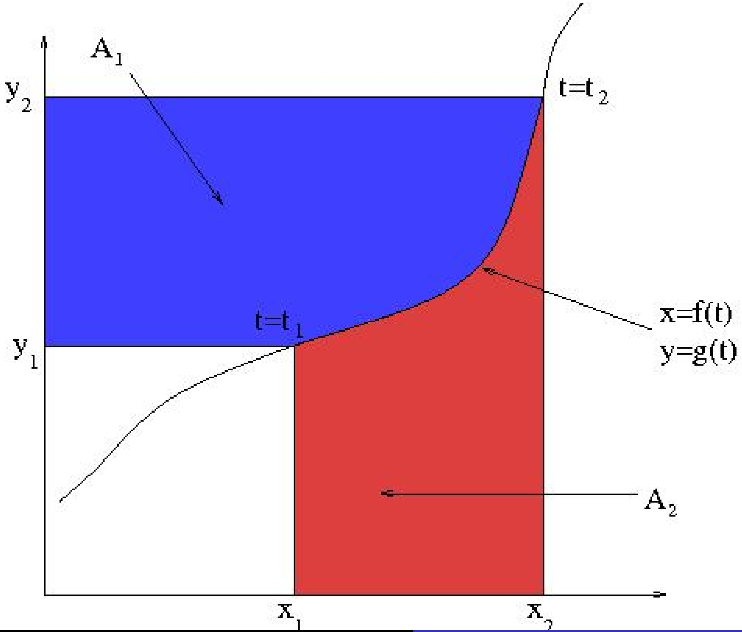
Inverse functions: f is $1-1$ with inverse f^{-1}

$$\begin{aligned}\int f(x)\,dx &= xf(x) - \int xf'(x)\,dx \\ &= xf(x) - \int f^{-1}(f(x))f'(x)\,dx \\ &= xf(x) - \int f^{-1}(u)\,du\end{aligned}$$

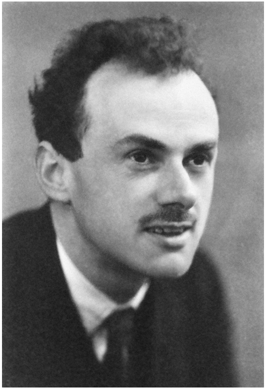
Integration by Parts implies Taylors Formula

$$
\begin{aligned}
f(x) &= f(0) + \int_0^x f'(x-t)\,dt \\
&= f(0) + xf'(0) + \int_0^x tf''(x-t)\,dt \\
&= f(0) + xf'(0) + \frac{x^2}{2} f''(0) + \int_0^x \frac{t^2}{2} f''(x-t)\,dt
\end{aligned}
$$

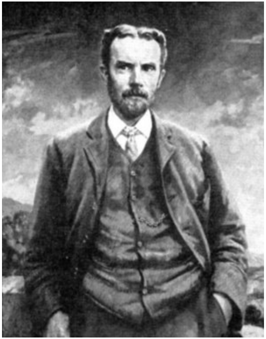
Picture for Integration by parts



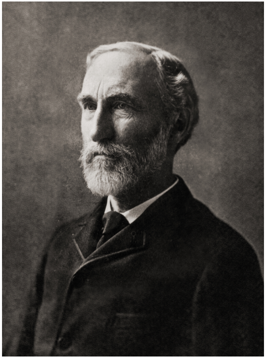
Mathematical Physicists



Paul Dirac (1902-1984) 1933 Nobel Prize



Oliver Heaviside (1850-1925) 1922 Faraday Medal Vector form of Maxwell's Equations



J. Willard Gibbs (1839-1903) 1863 1st American PhD in Engineering 1901 Copley Medal

Dirac Delta Function

Also called the unit impulse function.

$$\delta(x) = \begin{cases} 0 & \text{if } x < 0 \\ \infty & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases}$$

with the additional property of “unit mass”

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\text{so for nice } \phi, \int_{-\infty}^{\infty} \phi(x)\delta(x)\, dx = \phi(0)$$

Heaviside Step Function

Also called the unit step function

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

The value at $x = 0$ is not important. The choice of $1/2$ is from Fourier series, the Fourier series expansion for $H(x)$ converges to $1/2$ at $x = 0$.

The Physics is clear

A unit impulse of Force to a unit mass at rest at time 0 changes the velocity from 0 to 1.

$$F = ma(t) = \delta(t) = v'(t)$$
$$v(t) = H(t) = \int_{-\infty}^{t} \delta(t)\, dt$$

The value of the velocity at time $t = 0$ is not important.

The Mathematics is not clear

The function $\delta(x)$ is not a function. It only makes sense inside an integral. And even then, integrals should not depend on the value at one point. (ie

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

if $f = g$ except at one (or finitely many) point(s).

The Space of Test Functions

Generalize functions $g(x)$ are objects that make sense when integrated with a test function, $\phi(x)$, a function with infinitely many derivatives and compact support. Compact support means there is $[a, b]$ so that $\phi(x) = 0$ if $x \leq a$ or $x \geq b$. A classic example

$$\phi(x) = \begin{cases} \exp(-1/(1-x^2)) & \text{if } x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

Parts to the Rescue

Define the derivative of $f(x)$ of any function that makes sense under the integral by parts:

$$\int \phi(x) f'(x)\, dx = -\int \phi'(x) f(x)\, dx$$

Note the $\phi(x)f(x)$ term vanishes, since ϕ has compact support. (ie, it is zero at $\pm\infty$)

$$\begin{aligned}\int \phi'(x) H(x)\, dx &= \int_0^\infty \phi'(x)\, dx \\ &= \phi(x)\big|_0^\infty \text{ by the fundamental theorem of calculus} \\ &= \phi(\infty) - \phi(0) = -\int \phi(x)\delta(x)\, dx\end{aligned}$$

Hence $H'(x) = \delta(x)$.

More derivatives

The derivative of $\delta(x)$ satisfies

$$\int \phi(x)\delta'(x)\,dx = -\int \phi'(x)\delta(x)\,dx = -\phi'(0)$$

Let $R(x)$ be the ramp function

$$R(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

We have $R'(x) = H(x)$ since

$$\int \phi(x)R'(x)\,dx = -\int \phi'(x)R(x)\,dx = -\int_0^\infty x\phi'(x)\,dx$$

$$= -x\phi|_0^\infty + \int_0^\infty \phi(x)\,dx = \int \phi(x)H(x)\,dx$$

Where is the Error?

$$\int \frac{1}{x}\,dx = \frac{1}{x}x - \int x\left(-\frac{1}{x^2}\right)\,dx = 1 + \int \frac{1}{x}\,dx$$

Therefore $1 = 0$

Jokes?

Explain xkcd? http://www.explainxkcd.com/wiki/index.php/1201:_Integration_by_Parts

Integration by parts song

http://math-fail.com/2012/01/i-integrate-by-parts-total-eclipse-of-the-heart-pa
html

you can buy integration by parts tee shirts

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