Honors Day 2015
Archimedes Balancing Calculus

Steven F. Bellenot

Department of Mathematics
Florida State University

Honors Day
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Historical Time Line

1906 Palimpsest discovered

Rigorous development of Calculus

- 1815-1897 Weierstrass – unif conv
- 1789-1857 Cauchy – epsilons ε and deltas δ

Invention of Calculus

- 1646-1716 Leibniz 1685
- 1643-1727 Newton 1671/pub 1736

1598-1647 Cavalieri Indivisibles 1627/pub 1635

1229 Palimpsest Erased Christian text overwritten

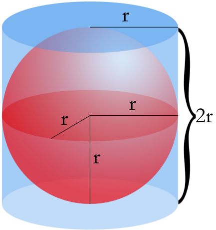
tenth century Palimpsest Greek copy created

287 BCE-212 BCE Archimedes

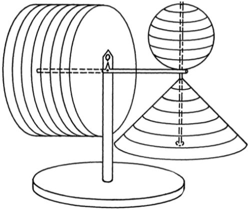
325 BCE-265 BCE Euclid

Archimedes and the Method

Fields Medal



Archimedes Grave-stone



Physical Model of the Method (1955)

The center of gravity

The left hand side of the balance is a cylinder of radius $2R$ and height $2R$ with a center of gravity at R to the left of the fulcrum. The cylinder has volume (and with a constant density, weight) of

$$2R\pi(2R)^2 = 8\pi R^3 \text{ balanced at } -R$$

Physics says we can treat a big object as a point with the same mass at the center of gravity. The center of gravity of the sphere and the cone are at $2R$. The weight of the cone is

$$(1/3)2R\pi(2R)^2 = 8\pi R^3/3$$

Solid geometry in the balance

$$R8\pi R^3 = 2R(8\pi R^3)/3 + 2RV$$
$$8\pi R^3 - (8\pi R^3)(2/3) = 2V$$
$$(8\pi R^3)(1/3) = 2V$$
$$\pi R^3(4/3) = V$$

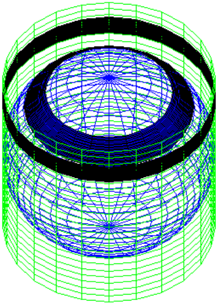
The volume of the sphere is $2/3$ of the circumscribed cylinder. This cylinder has volume $(2R\pi R^2 = 2\pi R^3)$.

Also it works for Surface Area

The area of the sphere is $4\pi R^2$ is the same as the lateral area of the circumscribed cylinder (perimeter $2\pi R$ times height $2R$.) Adding the top and bottom circles (each πR^2) gives the cylinder area $6\pi R^2$.

The area of the sphere is $2/3$ of the circumscribed cylinder.

The same area on cylinder and sphere



The area between h_0 and h_1 on the sphere and the cylinder are equal.

The Palimpsest

It had 177 pages, a palimpsest is a document written on recycled parchment.
10th century Byzantine Greek copy of mathematical works partially erased and overwritten with a Christan religious text by 13th century monks. It contains half a dozen works of Archimedes and the only know copy of the method. It also included works of others.
J.L Heiberg, a great scholar of primary sources for classical Greek geometry, recognized a transcript of some of the under text, from only one page, as the work of Archimedes.
The method is also called the method of mechanical theorems

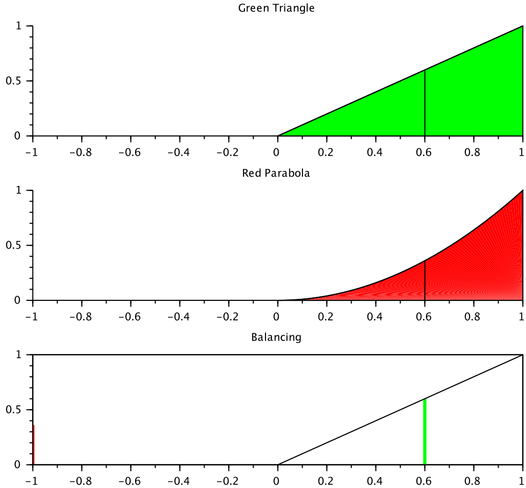
Simple example of the method

We will use the method to show

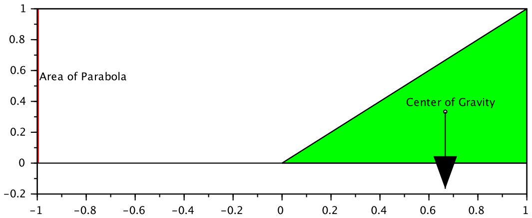
$$\int_0^1 x^2\,dx = \frac{1}{3}$$

We start by showing the area of the parabola $y = x^2$ above $[0, 1]$ is $2/3$ of the area of the triangle $y = x$ above $[0, 1]$

Example part 2

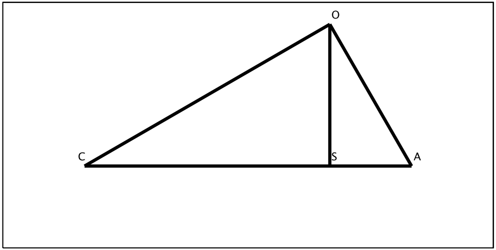


Example part 3



Since the center of gravity of the triangle is at $x = 2/3$
The area of the parabola = (2/3) the area of the triangle = (2/3)(1/2)=1/3

Similar Right Triangles

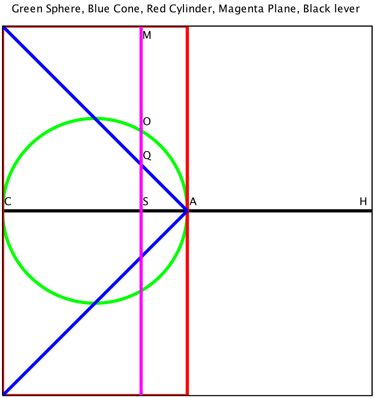


CAO is similar to *SAO* so hypotenuses are in ratio to short legs

$$\frac{CA}{OA} = \frac{OA}{SA}$$

$$OA^2 = CA * SA$$

The Diagram


Green Sphere, Blue Cone, Red Cylinder, Magenta Plane, Black lever
M
O
Q
C
S
A
H

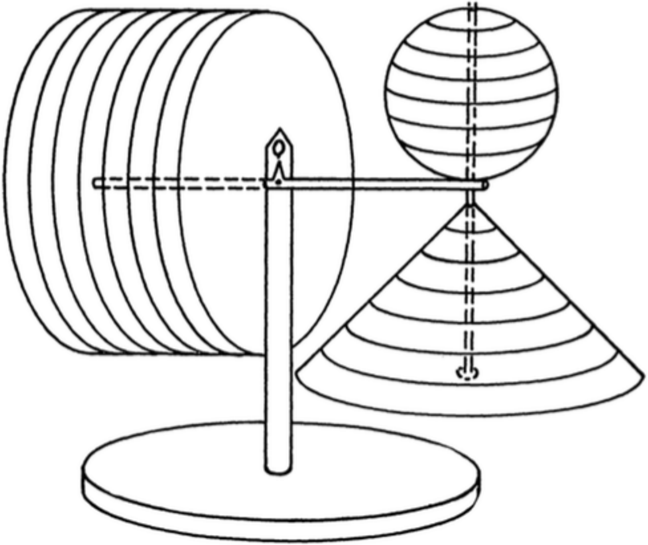
The method

$$\begin{aligned} MS = AC \text{ and } QS &= AS \\ MS * SQ &= CA * AS \\ &= AO^2 \text{ similar triangles } AOC \sim AOS \\ &= OS^2 + SQ^2 \text{ Pythagoras} \end{aligned}$$

$$\begin{aligned} \frac{HA}{AS} &= \frac{CA}{AS} \text{ Since } HA = AC \\ &= \frac{MS}{SQ} \\ &= \frac{MS^2}{SQ * MS} \text{ multiply by } MS \\ &= \frac{MS^2}{OS^2 + SQ^2} \end{aligned}$$

$$HA * (\text{circle in sphere + circle in cone}) = AS * (\text{circle in cylinder})$$

QED



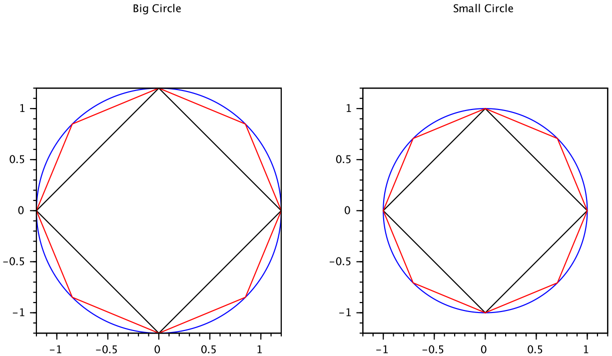
Book I has 44 Propositions (9 more in Book 2)
Prop 33 $A_{\text{sphere}} = 4\pi R^2$
Prop 34 $V_{\text{sphere}} = (4/3)\pi R^3$

Proving $A = B$

To prove $A = B$, an analyst proves both $A \leq B$ and $B \leq A$.
To prove $A \leq B$, an analyst proves $\forall \varepsilon > 0,\ A \leq B + \varepsilon$
To prove $A = B$, an ancient geometer, shows that assuming $A < B$ leads to a contradiction and assuming $B < A$ leads to a contradiction.

Exhaustion

Euclid, Book 10, Proposition 2: The areas of two circles are in the same ratio as the squares of their diameters.



The polygons Exhaust the area of the circles.

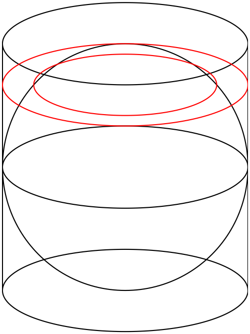
The area of similar polygons are in the same ratio as the squares of the diameters of the circle.

Cavalieri and Indivisibles

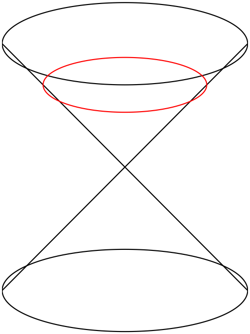
If two solids are included between a pair of parallel planes and the areas of the two sections cut by them on any place parallel to the including planes are always in a given ratio, then the volumes of the two solids are also in this ratio.
The solid is the union of the indivisible sections.

Cavalieri 2

Sphere inside Cylinder



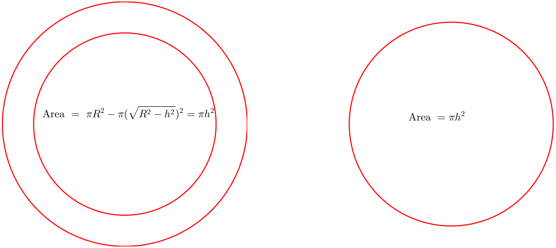
2 Cones



Cavalieri 3

Outer radius $= R$, Inner radius $= \sqrt{R^2 - h^2}$

radius $= h$



Cavalieri 4

Volume of Cylinder = Volume of Sphere + 2 * Volume of Cone

$$(2R)(\pi R^2) = V + 2((R)(\pi R^2)(1/3))$$
$$2\pi R^3 = V + (\pi R^3)(2/3)$$
$$V = (4/3)\pi R^3$$

Ballnot = 2 Pointy little cones.

Rotate $y = \sqrt{R^2 - x^2}$ about the x-axis.

$$V = 2\int_0^R \pi y^2\,dx$$

$$\begin{aligned} V &= 2\pi \int_0^R R^2 - x^2\,dx \\ &= 2\pi\,\left(R^2x - x^3/3\right)\Big|_0^R \\ &= 2\pi(2/3)R^3 = (4/3)\pi R^3 \end{aligned}$$

Rotate $y = \sqrt{R^2 - x^2}$ about the x-axis.

$$y' = \frac{-x}{\sqrt{R^2 - x^2}}$$

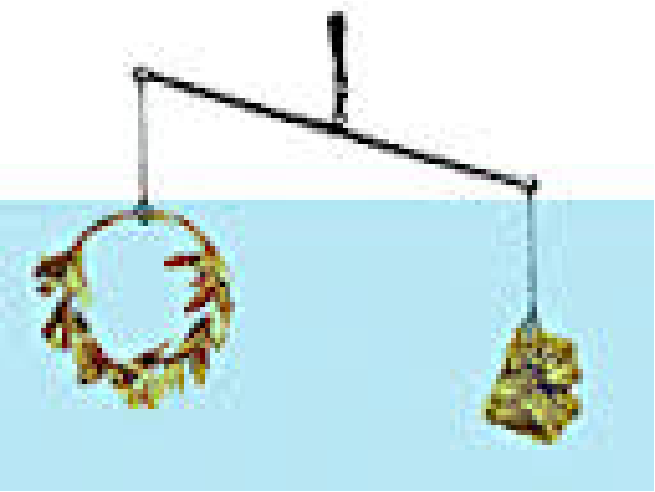
$$1 + y'^2 = \frac{R^2}{R^2 - x^2}$$

$$A = 2\int_0^R 2\pi y \sqrt{1 + y'^2}\, dx$$

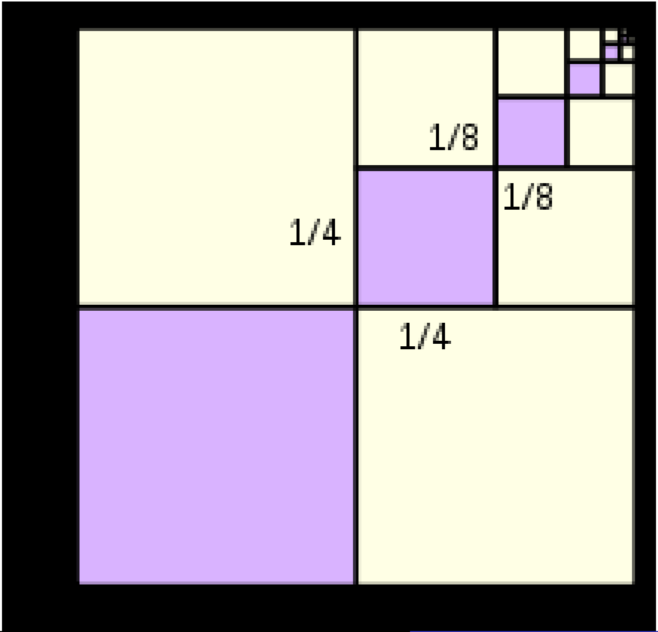
$$A = 4\pi \int_0^R R\, dx = 4\pi Rx|_0^R = 4\pi R^2$$

$$
\begin{aligned}
V &= 8\int_0^R \int_0^{\pi/2} \int_0^{\pi/2} \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho \\
&= 8\int_0^R \rho^2 d\rho \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi \, d\phi \\
&= 8(R^3/3)(\pi/2)(-cos(\pi/2) + cos(0)) = (4/3)\pi R^3
\end{aligned}
$$

Eureka!



The Series $\sum_1^\infty 4^{-n} = 1/3$



Jokes?

- What is an Archimedes Screw? The lack of credit he got for integration?
- Eureka! Archimedes replied Buoyantly.
- Last words? Do not disturb my circles.
- Jacob Bernoulli's Gravestone, the wrong spiral.