Honors Day 2018
Move over Honest Functions, Fourier leads the Bizarre

Steven F. Bellenot

Department of Mathematics
Florida State University

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Dr Kathleen Clark offers this fall 2018 Math Class that satisfies the upper division writing requirement.
(Its real number is in the works.)

You give an honest function, a bad name

Poincaré wrote in 1899

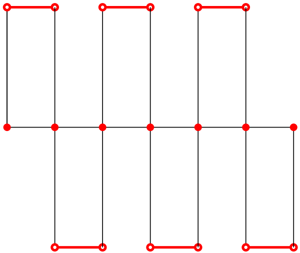
> *For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. . . . Formerly, when a new function was invented, it was in view of some practical end. Today they are invented on purpose to show that our ancestor's reasoning was at fault, and we shall never get anything more that that out of them. If logic were the teacher's only guide, he would have to begin with the most general, that is to say, the most weird functions.*

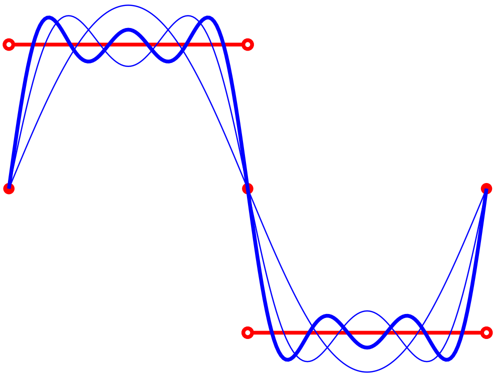
Fourier gets the blame, credit?

It is the distinction between a function and its representation(s).
1805, discontinuous functions could be represented by a Fourier series. That is a representation of nice trig functions could lead to something unexpected, namely discontinuity.
One of the features of the wave equation is discontinuous solutions.
Fourier moved us away from analytic expressions. (Here analytic means a local power series representation.)

Square wave

$$f(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$$





Modern Definition of Function

1923 Goursat: One says y is a function of x if to a value on x corresponds a value of y. One indicates this correspondence by $y = f(x)$.

1945 Black box analysis. The flight recorder 1953.

1960 Suppes: f is a function $\iff$ f is a relation and $\forall x \forall y \forall z, x\, f\, y$ and $x\, f\, z \implies y = z$

So it is a function machine, a black box, you put in a value and you get out a unique value. There is no required formula, graph, table or words. One doesn't worry about cause and effect or any law.

Earlier Attempts

Newton and Leibniz talked about curves.

- 1673 Leibniz . . . perform some function. First use?
- 1694 Bernoulli (Johann) . . . a quantity somehow formed from indeterminate and constant quantites.
- 1748 Euler A function of a variable quantity in an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.

Note, Euler's function had an expression, a formula.

Allowable formula's

All the formula's gave differentiable functions, actually analytic expressions. Exponentials, logarithms, trigonometric, infinite series, infinite products and infinite continued fractions.

Euler Discontinuous

A function that required two or more analytic functions. Like

$$y = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

1844 Cauchy noted this function was expressed by the single formula $\sqrt{x^2}$

Kinds of Functions: TGIF

The calculus reform movement of the 90's introduced *the gang of four* (Not the 70's band, nor the 60's Chinese political fraction, but the ways a calculus student should be able to view function examples:

Tables Babylonians (1800-1600 BCE) had tables of squares, cubes, and reciprocals.

Graphs Analytic Geometry and coordinates developed around 1640. Curves rather than function were the inital focus.

In words (Verbally) Babylonian templates. "I added twice the side to the square, the result is 10300. What is the side?"

Formulas Ptolemy (85-165) computed chords of a circle, trigonometry in the Almagest.

Time line

1350 Oresme – one quantity depends on another
1673 Leibniz – first use of the word function
1694 Bernoulli – formed from indeterminate and constants
1734 Euler – notation f(x)
1748 Euler Book– analytic expression of variable and constants
1755 Euler Book – f(x) notation
1805 Fourier – discontinous fuctions represented by Fourier Series
1821 Cauchy – Book explicit and implicit functions
1822 Fourier – not subject to a common law
1837 Dirichlet – f(x) continuous in the modern sense
1838 Lobachevsky – function definition requires continuity

1806 Ampère's Theorem

A continuous functions have derivatives everywhere but a finite number of points.
Based on continuous functions have increasing, decreasing or flat regions.
Continuous function had not yet been defined.
1878 Dini If a continuous function f has the property that f(x) + Ax + B possesses a finite number of maxima and minima (so it is piecewise monotonic) for all but a finite number of A's, then $f'(x)$ exists on a dense set.

1837 Dirichlet's Salt and Pepper function

$$y = \begin{cases} 0 & x \text{ rational} \\ 1 & x \text{ irrational} \end{cases}$$

Every open neighborhood contains both rational and irrational points. So it is nowhere continuous.
But this function is equal to one almost everywhere, which is constant.

It’s the law

Euler: Made the trig "lines connected with the circle" into functions. Allowed infinite series

Dirichlet: Difference between a function and its representation

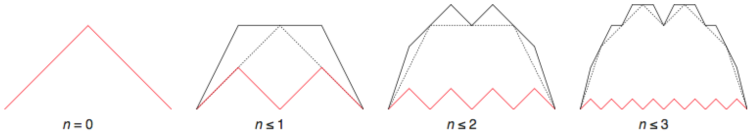
Fourier: No common law requirement

Weierstrass Monster

$$\omega(x) = \sum_{n=0}^{\infty} \frac{\sin(2^n x)}{2^n}$$
$$= \sin x + \frac{1}{2}\sin 2x + \frac{1}{4}\sin 4x + \dots$$
$$\omega'(x) =? \sum_{n=0}^{\infty} \cos(2^n x)$$
$$= \cos x + \cos 2x + \cos 4x + \dots$$

$\omega'(x)$ exists as a generalized function or distribution. Schwartz got the fields medal in 1950 for his work on the theory of distributions.

Takagi Monster 1903



A quote from Émile Picard (1856-1941)

If Newton had known about such functions he would have never created calculus.

Baire Category Result

In the set of continous functions on $[0, 1]$, the set of nowhere differential functions is of the 2nd category, in particular, the set is a dense G_δ.

The Baire category theorem was proved in Baire 1899 doctorial thesis. The result above is the Banach-Mazurkiewicz Theorem (1931?).

Calculus: A Pump not a Filter

This 1987 book was a snapshot of what was right and wrong with Calculus instruction at the time.
Use technology because it makes real world problems tractable.
(Friedler Calculus in the US 1940-2004):
There are some people who are still trying to deny the effect of technology on teaching calculus. These people will eventually retire.
Calculus Enrollments declined from 647K in 1990 to 538K in 1995.
Use technology to grade, flip or enhance – nothing about tractable.