Honors Day 2019
The Delian Double Down
Ancient Greek Solutions not Euclid legal

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The Delian problem

The year was around 430 BCE and the plague was real, about a quarter of the population died from it.

Why were the gods angry and how to appease them? The oracle required no less than Plato (428-348 BCE) to decrypt it.

The altar, double the existing one needed to be constructed. To shame the greeks for their neglect of mathematics and their contempt of geometry.

$\sqrt[3]{2} \approx 1.259921\ldots$

Since the volume of a cube is $V = s^3$, doubling the side s would produce a cube with $8 = 2^3$ times the volume. This is not the same as multiplying the side by $\sqrt[3]{2}$ which doubles the volume since $(\sqrt[3]{2})^3 = 2$.
For later, we note that this is a root of the polynomial $x^3 - 2$.

The Delian problem required constructing the length $\sqrt[3]{2}$.

Greek Statement

Hippocrates (470-410 BCE) used ratios and express this problem as finding two mean proportion between two lengths.

That is given a, b find x, y so that $a : x = x : y = y : b$ or

$$\frac{x}{a} = \frac{y}{x} = \frac{b}{y}.$$

With modern understanding

$$(\frac{x}{a})^3 = \frac{x}{a}\frac{y}{x}\frac{b}{y} = \frac{b}{a}$$

(The Elements notes the mean proportion $a : x = x : b$ gives the square root.)

A three dimensional solution

Archytas (428-350 BCE) found the solution as the intersection of three surfaces, with modern understanding

1. a Cylinder $x^2 + y^2 = ax$
2. a Torus $x^2 + y^2 + z^2 = a\sqrt{x^2 + y^2}$
3. a Cone $x^2 + y^2 + z^2 = a^2 x^2 / b^2$

If $P(p, q, r)$ is the point of intersection, and $N(p, q, 0)$ is the projection onto the xy-plane.
$|OP| = \sqrt{p^2 + q^2 + r^2}, |ON| = \sqrt{p^2 + q^2}$

$$\frac{a}{\sqrt{p^2 + q^2 + r^2}} = \frac{\sqrt{p^2 + q^2 + r^2}}{\sqrt{p^2 + q^2}} = \frac{\sqrt{p^2 + q^2}}{b}$$

A solution using conics

Menaechmus (380-320 BCE), perhaps creating conics.

In modern notations, given a, b let x, y be the (non-zero) intersection of

1. $y^2 = bx$, a rightward parabola
2. $x^2 = ay$, an upward parabola

From the first $\frac{b}{y} = \frac{y}{x}$ and the second $\frac{a}{x} = \frac{x}{y}$, hence

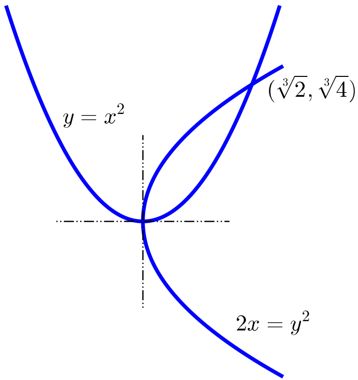
$$\frac{a}{x} = \frac{x}{y} = \frac{y}{b}$$

Descrates (1596-1650) added the equations, getting a circle

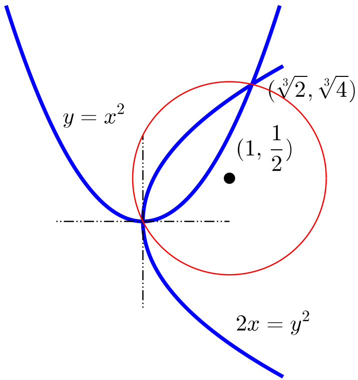
$$x^2 + y^2 - bx - ay = (x - \frac{b}{2})^2 + (y - \frac{a}{2})^2 - \frac{a^2 + b^2}{4} = 0$$

Thus a single parabola, plus straightedge and compass suffices.

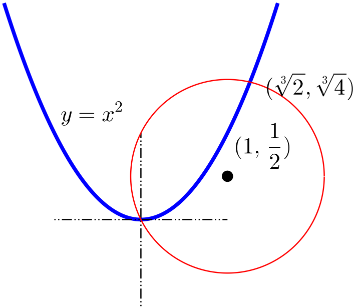
Menaechmus Graph



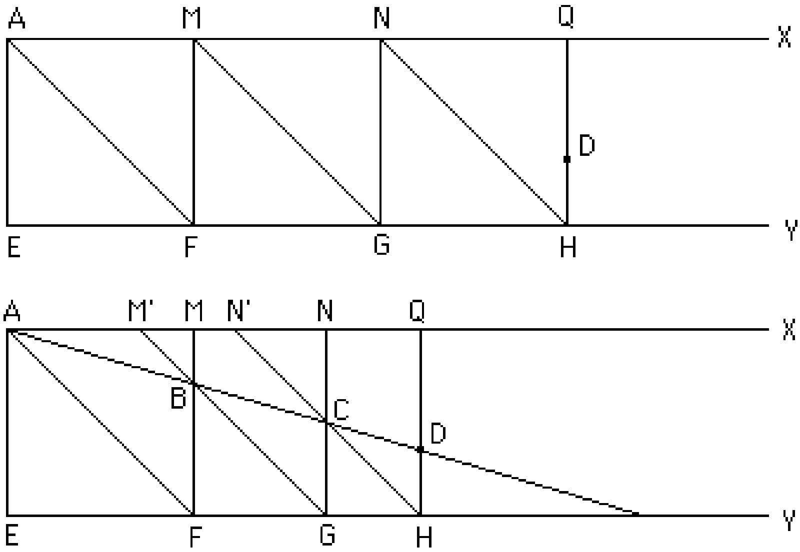
Menaechmus Graph Descrates Circle



Single Parabola + Euclid



Mechannical solution: Plato's Machine



Mechannical solution: Plato's Machine

$$\triangle ABF \sim \triangle BCG \sim \triangle CDH$$

$$\frac{BF}{AF} = \frac{CG}{BG} = \frac{DH}{CH}$$

$$\triangle AFE \sim \triangle BGF \sim \triangle CHG$$

$$\frac{AF}{AE} = \frac{BG}{BF} = \frac{CH}{CG}$$

Multiplication yields the desired

$$\frac{BF}{AE} = \frac{CG}{BF} = \frac{DH}{CG}$$

Euclid Construction Tools

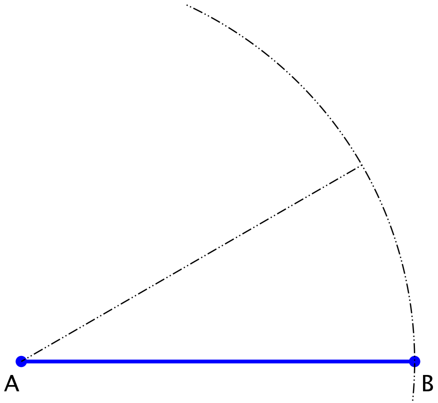
These are often called ruler-and-compass constructions:

- Actually the ruler has no markings, it is just a straightedge. One can mark a line between any two points. (Or extend an existing line.) Today's construction uses a chalk line, which is snapped.
- The compass is collapsing, it loses its setting when moved. One can mark a circle centered on one point and passing through another. Today's compasses can retain their settings and can be used to transfer lengths.

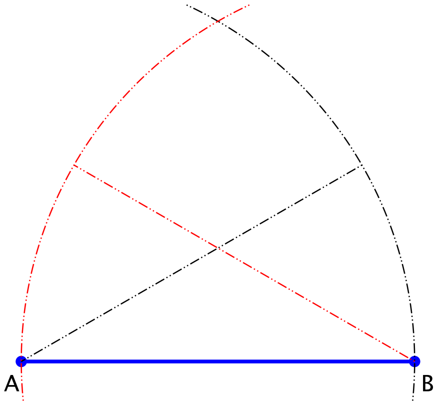
The Elements (circa 300 BCE), by Euclid, are the source of these rules

Construction 1: Given AB

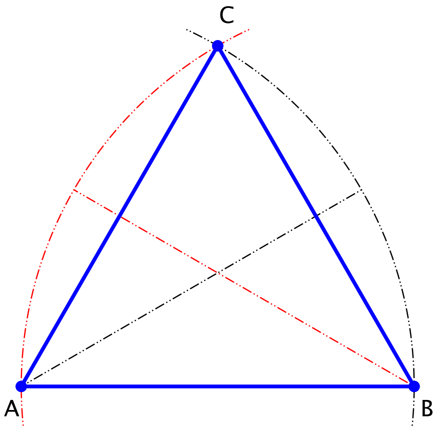
Construction 1: Circle AB



Construction 1: Circle BA



Construction 1: Equilateral Triangle ABC



Construction Types

The greeks divided constructions from best to worst

Planar Straight edge and compass – the ideal.

Solid Also used conic sections (any?)

Linear The other class. For example, construction of special curves or use of the Neusis – described later.

Classic Greek Unsolved Problems

Can you using only a compass and an unmarked straightedge:

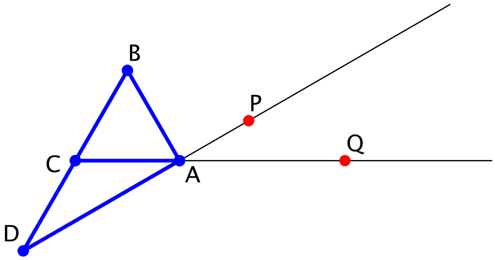
- Doubling a cube (impossible Wantzel 1837)
- Trisecting an angle (impossible Wantzel 1837)
- Square a circle (impossible Lindemann 1880, π transcendental)

There are crazy "circle-squarers" and "angle-trisectors" to this day. The third group, "cube-doublers" were more popular in ancient Greek times.

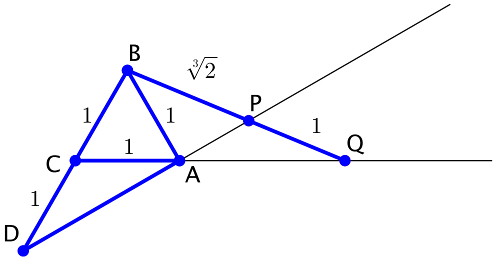
Neusis – a not inconsiderable error

Pappus of Alexandria (ca 325 AD) quote. If all else fails one can use a Neusis. One can think of a Neusis as a straightedge with two marks on it say P and Q. The goal is to draw the line through a fixed B (the pole) so that P is on one curve (the catchline) and QB is on another (the directrix) while $|PQ|$ is the fixed distance (the diastema).
Archimedes and Issac Newton both used the neusis.

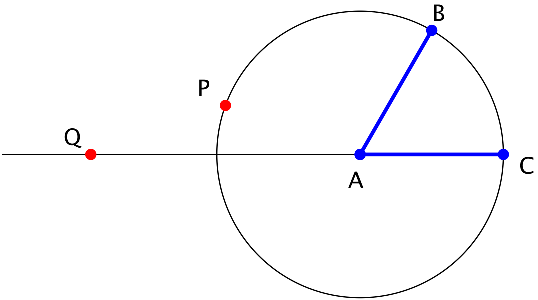
Neusis Doubling I



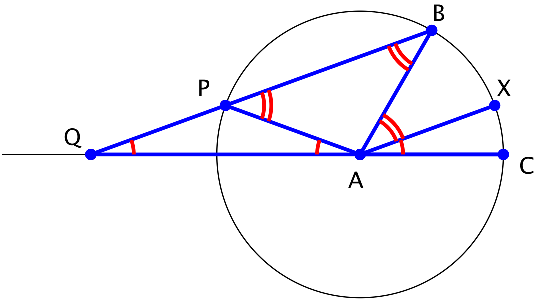
Neusis Doubling II



Neusis Trisecting I



Neusis Trisecting II



Constructing Regular n-Polygons

Euclid $n = 3, 4, 5, 6, 15 = 3 \cdot 5$, If n, then $2n$.

Gauss $n = 17$ and Fermat primes of the form $2^{2^m} + 1$

Planar If and only if $n = 2^m p_1 p_2 .. p_k$ where each p_i is a distinct Fermat prime. (Only 5 known, so only 31 odd n). Not $7, 9, 11, 13, 14, 18, 19, 21, 22, 23, 25 \ldots$.

Wantzel 1837 these are the only constructable n-gons.

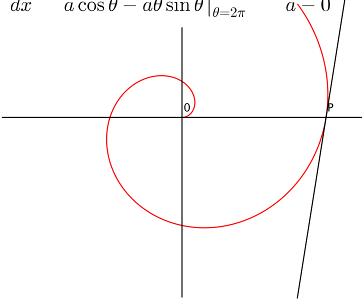
Trisector If and only if $n = 2^r 3^s p_1 .. p_k$ where the primes $pi = 2^t 3^u + 1$ are distinct Pierpoint primes. No for 11

Neusis Yes for $3 \leq n < 23$; No for 23, unknown for 25. One can solve some quintics that are not solvable by radicals.

Archimedes Spiral

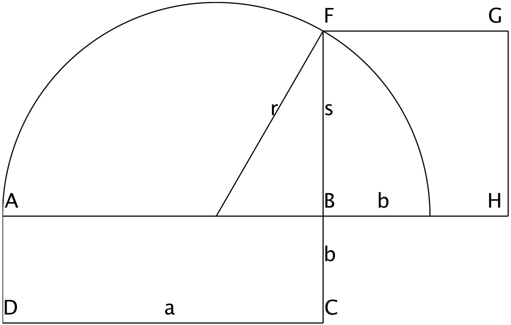
$$r = a\theta;\ x = r\cos\theta;\ y = r\sin\theta$$

$$\frac{dy}{dx} = \left.\frac{a\sin\theta + a\theta\cos\theta}{a\cos\theta - a\theta\sin\theta}\right|_{\theta=2\pi} = \frac{0 + a2\pi}{a - 0} = 2\pi$$



Triangle-squaring

$$r = (a+b)/2;\ s^2 = r^2 - (r-b)^2 = (a+b)^2/4 - (a-b)^2/4 = 4ab/4$$



Archimedes Spiral Trisecting