The Hydra

This Welcome is brought to you
by Gödel,
Hercules, and the Countable Ordinals

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Hercules and The Hydra

The Hydra was a monster with nine serpent heads. Each head cut off, was replaced by two new ones. One of the twelve tasks of Hercules, the second, was to kill the Hydra.

Logicans have turned the hydra into a mathematical game as we will see later.

This picture of the monster might remind us of the Florida Legistature and its ideas on education. Where are the other heads? Are they missing? My wife Ellen told me no, they're just too far to the right to be included in the frame.

The number of heads vary according to the story teller, as well as the number of replacement heads.

Politicians are not your friend

- Hours 1 – 144 cheap in-state tuition
- Hours 145 up expensive out-of-state tuition
- Changing majors might be costly
- They are bastards, courses dropped on the 5th day of classes count in the 144 hours

The politicians have come up with a new grand plan with the goal of making the cost of education cheaper for the state. The plan is called Excess Credit, which is near impossibly complex. For a math major who needs 120 hours to graduate, they are limited to 144 hours of the cheaper in-state tuition, any additional hours have to a the higher out-of-state rate. Exceptions are made for some credit by exmination (like CLEP) or for remedial work. The handling of credit for transfer students with AA degrees is completely different than before. I'm told that there were so many questions, that the politicians stopped answering them.

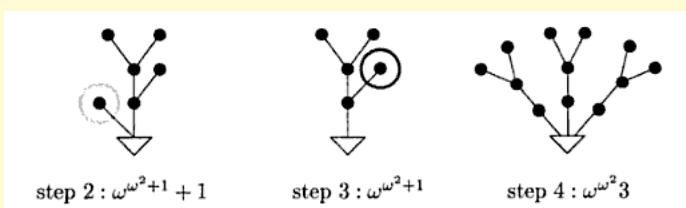
This could make changing your major very expensive, which seems not very student friendly. Students will likely be unaware of the looming expense at the end of their college experience. Note the importance of dropping before the end of drop/add.

Engineering majors require 128 hours, so their Excess Hours is different. Double majors also are a complication.

Since this talk, the Excess Credit hour limit has decreased twice. Instead of in-state vs out-of-state, the fees are computed with a surcharge instead. Is the Registrar's excess credit policy currently in effect.

Obvioysly this is not a mathematical tidbit. Unless one considers it as an example of changing the rules (or definitions).

Modeling the problem



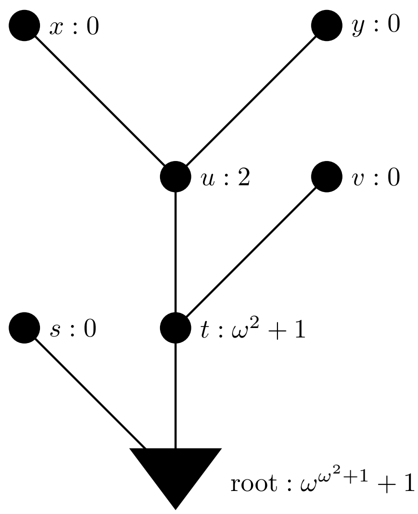
The Hydra game starts with a rooted tree. Any leaf can be chopped off. If the parent of the leaf is the root, nothing is added. (For example, in the picture the circled leaf in step 2, is just removed for step 3.) On the other hand, if parent p of the leaf is not the root, then the subtree rooted at p is replicated n times, (where n increases by one after each chopped off leaf,) each attached to p parent. (For example, in step 3, p is the vertex adjacent to the root, and 2 more copies of the Y shaped subtree are added to the root.)

The trees all can be identified with at countable ordinal, which is defined inductively as the sum of the ordinals based on the ordinals for each child subtree.

Can Hercules cut all the leaves off (leaving only the root)? If so then he wins this game.

At this point, I played with the Hydra game applet available at Andrej Bauer's blog

First we do some examples to compute the countable ordinals. We start with the figure below:



The countable ordinal 0 is given to each leaf, and if the non-leaf a has children with ordinals $\alpha_1 \geq \alpha_2 \geq \ldots \alpha_n$, then the subtree at a has ordinal

$$\omega^{\alpha_1} + \omega^{\alpha_2} + \ldots \omega^{\alpha_n}.$$

As might be expected $\omega^0 = 1$.

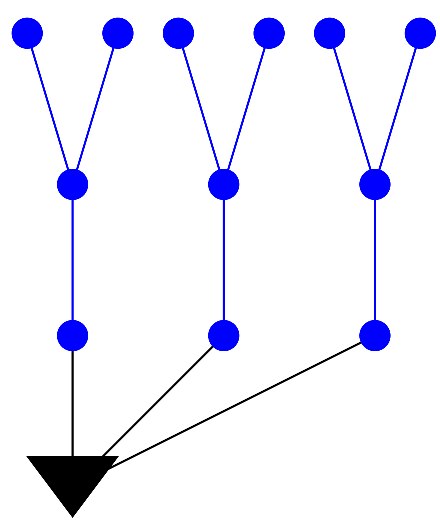
So the subtrees at vertices $x, y, v,$ and s all have ordinal 0. The subtree at u has ordinal $w^0 + w^0 = 2$. The subtree at t has ordinal $\omega^2 + \omega^0 = \omega^2 + 1$. The whole tree thus has ordinal

$$\omega^{\omega^2+1} + \omega^0 = \omega^{\omega^2+1} + 1.$$

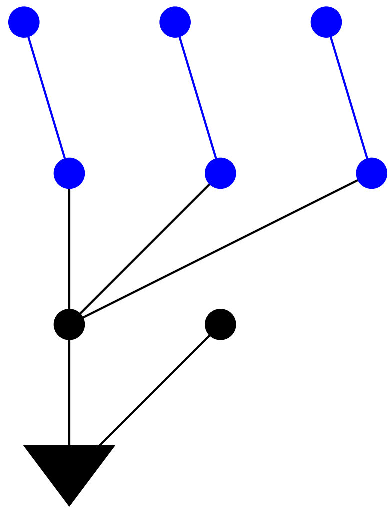
Ordinal arithmetic is non-abelian:

$$\omega + \omega^2 = \omega^2 \neq \omega^2 + \omega$$
$$3\omega = \omega \neq \omega + \omega + \omega = \omega 3$$

Which is why the tree in step 4 has ordinal $\omega^{\omega^2}3$ and not $3\omega^{\omega^2}$.



The hydra (above) that results when vertex v is chopped at step 3. Note the three blue Y-shaped subtrees.



The hydra (above) that results if vertex y is chopped instead of v. Note the three blue subtrees.

The Theorem

Theorem: You can't lose.

(An unprovable result ala Gödel)

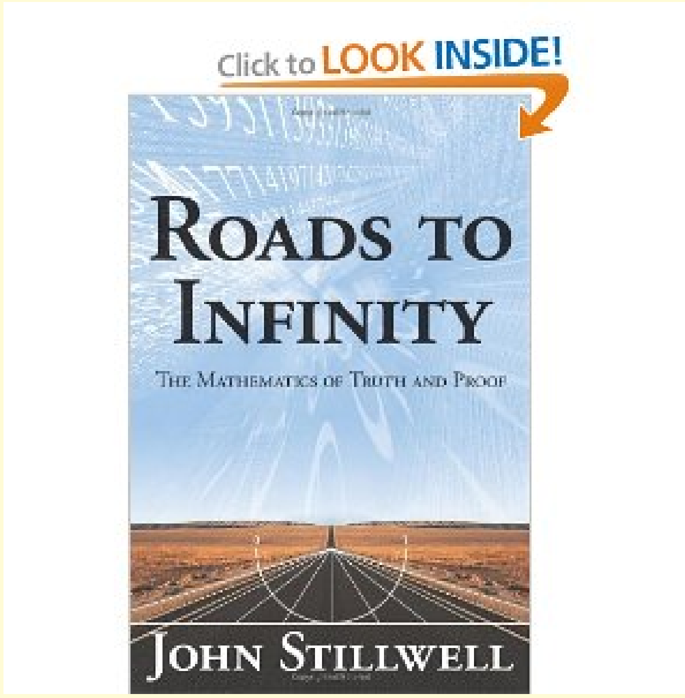
But it might take a long time.

Each time a leaf is chopped off, the number of vertices grows fast but the ordinal decreases. Since the ordinals are well ordered, evey decreasing sequence of ordinals is finite. So eventually Hercules wins.

This is a concrete example of a Gödel Theorem. A true statement in number theory (Peano Arithematic) which cannot be proved in number theory (Peano Arithematic).

Knowing about countable ordinals is not something most undergraduate students learn. Well ordering is more common knowledge. Of course, the Gödel result is well known, but often not a subject of study for undergraduates.

A reference



Stillwell's book is a popular book on set theory and logic that includes an entire section on the hydra. It also talks about the proof of the theorem.

At the time of this talk, the book had just come out. I got a copy at Mathfest 2010, the year it was published.

Why this topic?

I learned about countable ordinals from Halmos' Naive Set Theory while doing a self study as an undergraduate. One of my early, graduate student housemates, Fred Zemke, was the first to show me countable ordinals were important in gauging the strenght of certain axioms systems. His senior thesis at Reed Colege was on countable ordinals. Hilbert Levitz, a colleague at Florida State University told be about the hydra result. And of couse, Stillwells book had just come out.

It is very exciting to have a concrete example of a true theorem of number theory which cannot be proved in number theory. A Gödel statement if you like. Much more bang than the usual diagonal proof.

Countable ordinals are in MHF 5206 Foundations of Mathematics, which I have taught a number times, including summer 2008. Many courses cover well-orderings.

My period of self study was between semesters when Harvey Mudd was very quiet. Mel Henriksen came for an interview for the Mathematics Department Chair position. I was the only math major available for his meeting with the students. It was a great introduction into Mel's mathematical universe of people doing math. I was to blame for him being hired, he would say. I would reply, I had to recommend you, if only to get back the cigarettes he bummed from me. Both of us gave up this vice long ago.

Picture sources

Hydra picture is from Scary for kids.

Tree slide was not re discovered by the authors google searches.

Book Cover is from amazon.