The Principle of Maximal Ignorance and the Chord Paradox

This Welcome is brought to you
by Random Endpoints,
Throwing Sticks, and
Throwing Darts

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Paradox: 3 distinct ways of selecting a random chord

What is the probability that a random chord of a circle is larger than the side of the inscribed equilateral triangle?

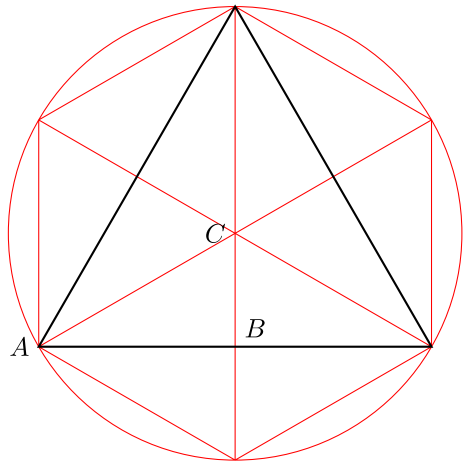
1. Random endpoints. Answer 1/3
2. Throwing sticks: Random radii. Answer 1/2
3. Throwing darts: Random midpoints. Answer 1/4

The answer is 1/3, if we use Method 1. This method picks the endpoints of the chord at random: two points on the circumference are selected.

The answer is 1/2, if we use Method 2. This method picks a radius at random and then a point on the radius at random. The chord is perpendicular to the radius at the point selected. Alternately one throws a large stick and the chord is the portion of the stick inside the circle (perhaps extending the stick).

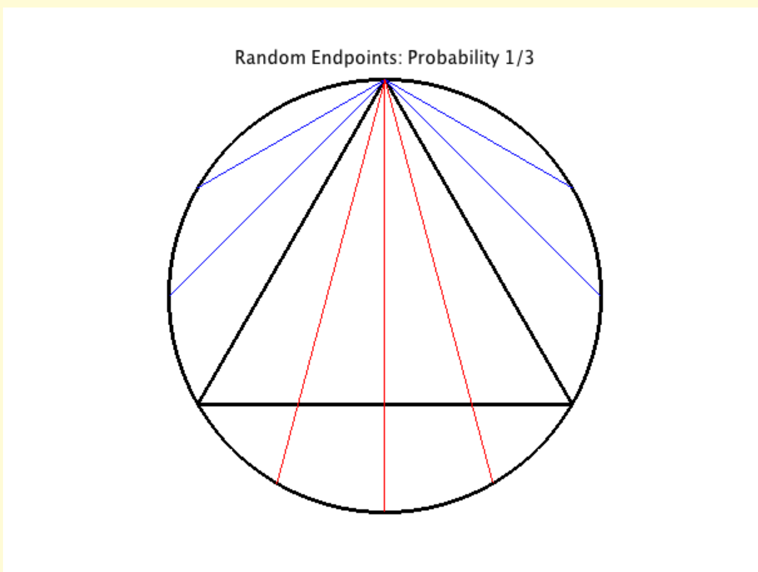
The answer is 1/4, if we use Method 3. This method picks a point inside the circle at random to be the midpoint of the chord. The chord is perpendicular to radius going through chosen point. Alternately, we can think of the point being picked by the throw of a dart.

For the slides below, we don't need to know the length of the side of inscribed equilateral triangle. On the other hand it is easy to use the figure below to compute it. If the radius is one, then since $\triangle ABC$ is a $30-60$ right triangle, the length of the side is $\sqrt{3}$.



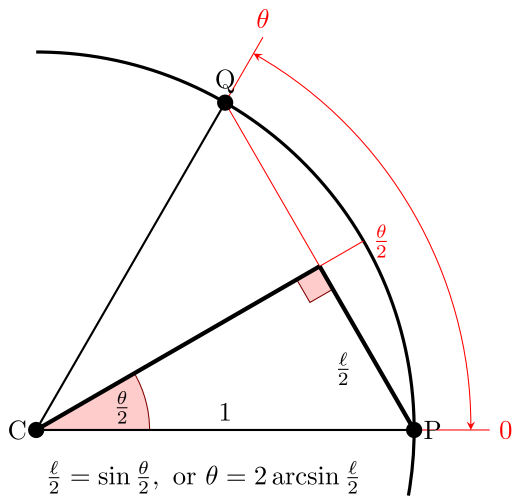
What about diameters? They don't matter, we can ignore them as they have zero probability. These methods differ on how they deal with diameters. Method 3 every diameter is selected, if and only if, the point is the center of the circle. Method 2 each diameter selected twice. Method 1 picks each diameter only once.

Random Endpoints



In the case of random endpoints, we can place our triangle with a vertex at the first selected point. If the second point is in the bottom 1/3 of the circumference (red chords) the chord has length larger than the side of the equilateral triangle. The blue chords have smaller lengths.

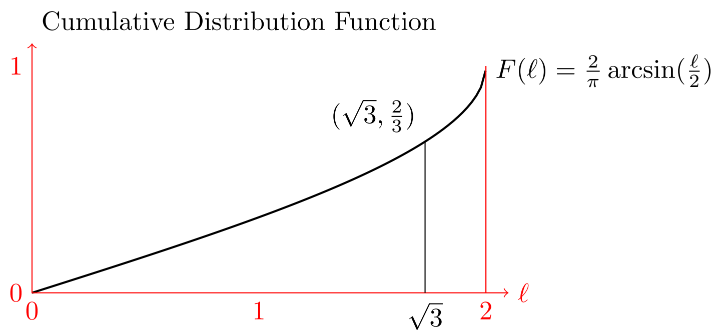
The original talk ran a Scilab program to simulate these probabilities. This program has been updated and is in `https://www.math.fsu.edu/~bellenot/talks/fsu08.13/chord.sce`. Instead of running the program, we will compute Probability densities. The following picture exposes the geometry of Method 1.



To compute the cumulative distribution function:

$$\begin{aligned} F(x) &= P(\ell \leq x) \\ &= P(\theta \leq 2 \arcsin \frac{x}{2}) \\ &= \frac{2}{\pi} \arcsin \frac{x}{2} \end{aligned}$$

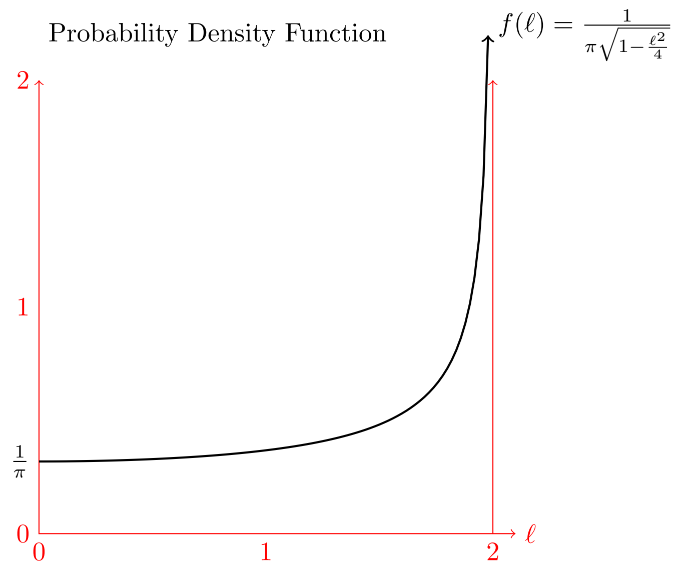
We divide by π, since $0 \leq \theta \leq \pi$, Since the length of the equilateral triangle is $\sqrt{3}$, we can read off the answer from the plot. Two thirds of the chords have shorter lengths



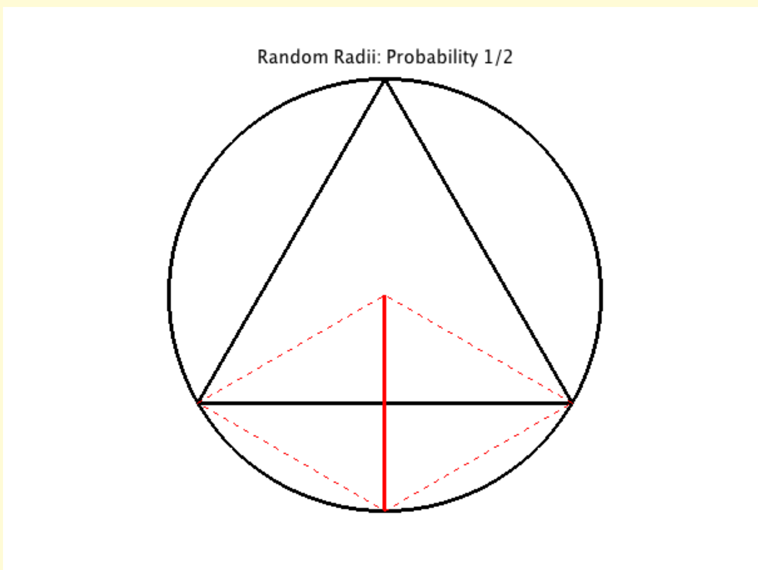
Taking the derivative of $F(x)$ gives the probability density function.

$$f(x) = \frac{1}{\pi\sqrt{1 - \frac{x^2}{4}}},$$

Note the asymptote at $\ell = 2$, and that the probability is almost uniform for values $\ell < 1$.

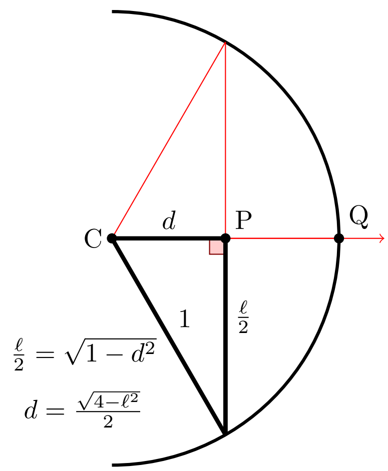


Random Radii



We orient the inscribed triangle so its base is perpendicular to the radius. Note that red dotted triangles that share the base side of the equilateral triangle are congruent. So half of the radius is on either side of the triangular base. The chord, is longer than a side, if the point is inside the triangle or 1/2 the time.

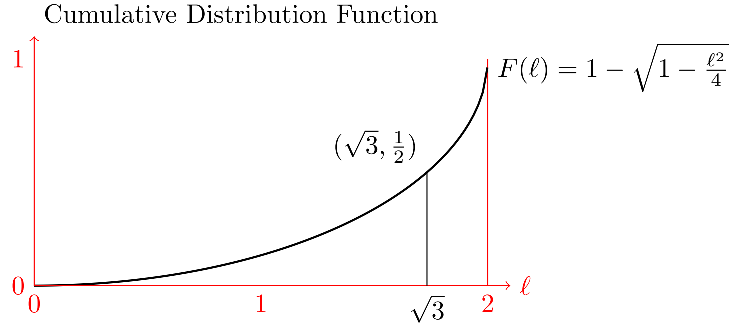
The following picture exposes the geometry of Method 2.



To compute the cumulative distribution function:

$$\begin{aligned} F(x) &= P(\ell \leq x) \\ &= P(1 - d \leq 1 - \sqrt{1 - \frac{x^2}{4}}) \\ &= 1 - \sqrt{1 - \frac{x^2}{4}} \end{aligned}$$

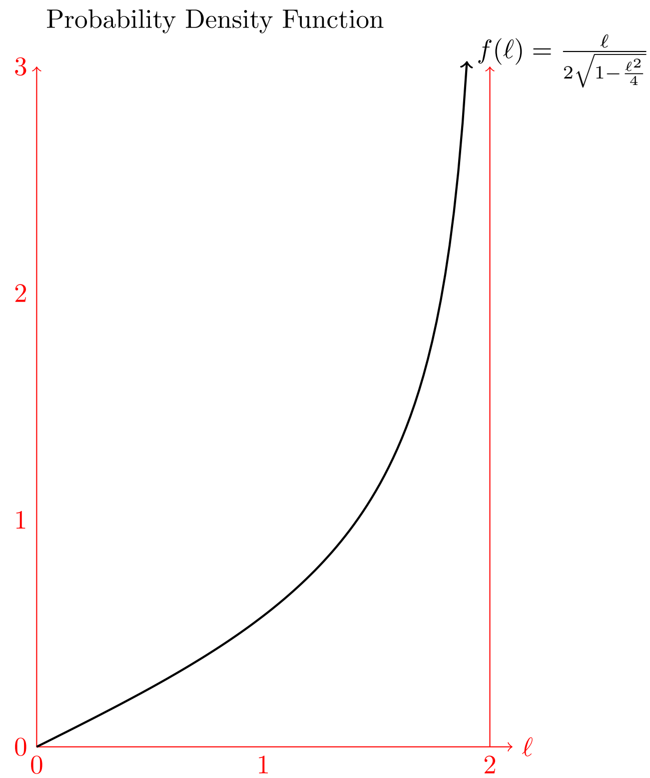
Since $1 - d$ is uniform. Since the length of the equilateral triangle is $\sqrt{3}$, we can read off the answer from the plot. One half of the chords have shorter lengths.



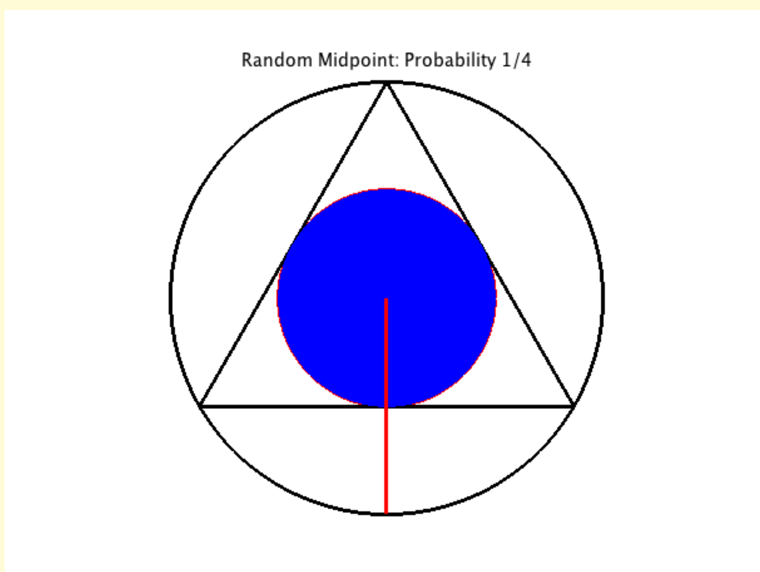
Taking the derivative of $F(x)$ gives the probability density function.

$$f(x) = \frac{x}{2\sqrt{1 - \frac{x^2}{2}}}$$

Note the asymptote at $\ell = 2$, and that the probability of short lengths is much less than uniform.

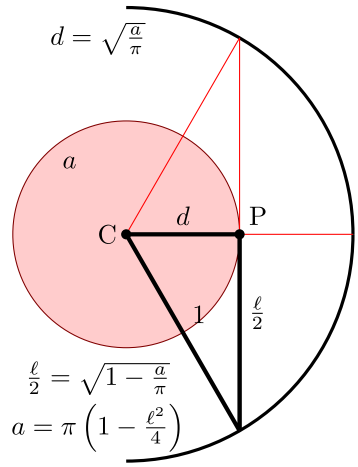


Random Midpoints



Consider the incircle of the triangle, Any midpoint outside the incircle will have a chord of length smaller than the triangles side. The blue area is 1/4 of the large circle as its radius is one half the side. So only the chord length is longer than the side of the triangle only 1/4 of the time.

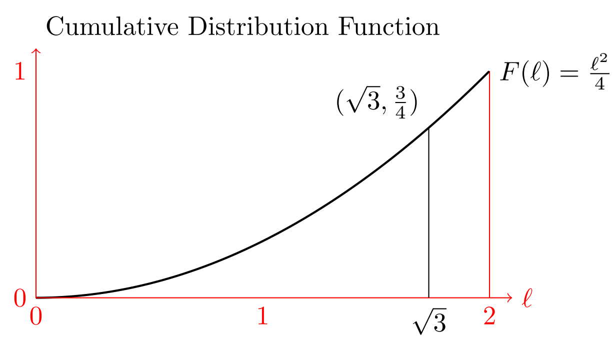
The following picture exposes the geometry of Method 3.



To computer the cumulative distribution function

$$\begin{aligned} F(x) &= P(\ell \leq x) \\ &= P(\pi - a \leq \pi \frac{x^2}{4}) \\ &= \frac{x^2}{4} \end{aligned}$$

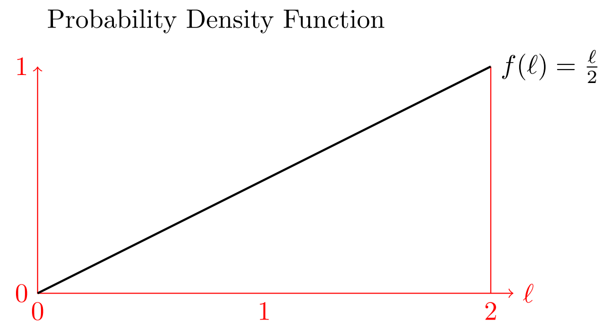
Since the area of the whole is π, we divide by π to get the probability. Since the side of the equilateral triangle has length $\sqrt{3}$, we can read off the answer from the plot. Three Quarters of the chords have shorter lengths.



Taking the derivative of $F(x)$ gives the probability density function.

$$f(x) = \frac{x}{2}$$

Note that the density function is bounded and that for small values the density is less than uniform.



Maximal Ignorance

The chord paradox is due to Joseph Bertrand (1889). Both Borel (1909) and Poincaré (1912) and 8 others comment before 1965.

Edwin Jaynes (1973) proposed his "maximal ignorance" solution. The solution must be translation and scale invariant, hence it is random radii.

The actual solution uses the integral equations of the in variance to derive the random radii solution.

This is not the slide I would use today. It turns out that Jaynes' solution isn't universally accepted. It is better to think that the question is not well-posed, and Jayne introduced *Maximal Ignorance* to change the question.

At the time of the talk, I didn't have the time to digest Jaynes. Instead, I had a joke slide which claimed maximal ignorance was shown by a calculus student whose answer to the question, what is the derivative of e^x, was xe^{x-1}.

In the late 1970's, I was advising an undergraduate transfer student who had completed the calculus sequence. We agree that ODE (Ordinary Differential Equations) will be the next course. But he quickly found it over his head. Jim Snover, the head of basic math at time, gave him a quick diagnostic test, and the student made the mistake above.

How was this possible? His Calculus 1 was done in high school and mainly did the calculus of polynomials. His Calculus 2, which he got an A, had so much extra credit that his grade reflected his willingness to work and not his understanding. His Calculus 3, which he got a C, was from a teacher who was not doing well. The teacher gave C's like candy, rather than the grades the students earned. The student was started back from almost the beginning at FSU.

Let's find the probability density function of the distance of the chord form the center of the circle under each of our methods. Let $b = \ell/2$ be half the length of the chord. The cumulative distribution function for b, call it $G(x)$, is related to $F(x)$, the cumulative distribution function for ℓ, by the formula:

$$G(x) = F(2x);$$

taking derivatives, gives us density functions:

$$g(x) = G'(x) = F'(2x) \cdot 2 = 2f(2x).$$

Now let $H(x)$ be the cumulative distribution function for d, the distance between the center of the circle and the chord. Since $d^2 + b^2 = 1$:

$$\begin{aligned} H(x) &= P(d \leq x) \\ &= P(d^2 \leq x^2) \\ &= P(1 - d^2 \geq 1 - x^2) \\ &= 1 - P(1 - d^2 < 1 - x^2) \\ &= 1 - P(1 - d^2 \leq 1 - x^2) \text{ by continuity} \\ &= 1 - P(\sqrt{1 - d^2} \leq \sqrt{1 - x^2}) \\ &= 1 - P(b \leq \sqrt{1 - x^2}) \\ &= 1 - G(\sqrt{1 - x^2}) \end{aligned}$$

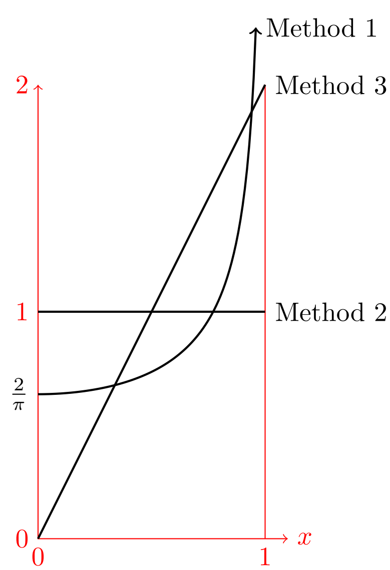
Taking derivatives gives density functions

$$\begin{aligned} h(x) &= -g(\sqrt{1 - x^2}) \cdot \frac{-2x}{2\sqrt{1 - x^2}} \\ &= \frac{xg(\sqrt{1 - x^2})}{\sqrt{1 - x^2}}. \end{aligned}$$

Density Functions: For chord length, $0 \leq x \leq 2$, otherwise $0 \leq x \leq 1$.

	Chord Length ℓ	Half Chord Length b	Distance From Center d
Method 1	$f(x) = \frac{1}{\pi\sqrt{1 - \frac{x^2}{4}}}$	$g(x) = \frac{2}{\pi\sqrt{1 - x^2}}$	$h(x) = \frac{2}{\pi\sqrt{1 - x^2}}$
Method 2	$f(x) = \frac{x}{2\sqrt{1 - \frac{x^2}{4}}}$	$g(x) = \frac{x}{\sqrt{1 - x^2}}$	$h(x) = 1$
Method 3	$f(x) = \frac{x}{2}$	$g(x) = 2x$	$h(x) = 2x$

Density Functions $h(x)$



Method 2, has the property that the distance from the center is uniform gives a reason for its preference.

Why this topic?

I don't remember when or where I learned these facts. It is a nice example to show that sometimes there are several correct answers. The list of three answers fit well with my format of three tidbits per lecture. While we have supplemented the talk with several pages of mathematics, the time requirement for the presented material is brief.

A fourth method, with a fourth answer, is to pick two points inside the circle at random and let the chord be determined by these two points, (perhaps, throwing both a red and a blue dart). This method is studied in a class web page linked here.

Picture sources

Slide pictures by the author using Scilab.

Other pictures by the author using TikZ.