The well posed student

This welcome is brought to you
by existence,
uniqueness,
and the backwards heat equation

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Jacques Hadamard (1902) Well-Posed Problems

Hadamard was famous for proving the prime number theorem in 1896. But we are interested in his idea of well-posed problems in PDE, Partial Differential Equations.

Vallée-Poussin independently proved the prime number theorem, that the number of primes less that $N, \pi(N) \sim N/\ln N$. Both lived long lives, Hadamard (1865-1963), Vallée-Poussin (1866-1962). Proving major theorems can add years to your life.

Jacques Hadamard 1902 notion of a well-posed problem

Context: Solutions to a PDE modeling a physical process.

1. The equation has a solution
2. The solution is unique
3. The solutions's behavior changes continuously with the initial conditions.

If it is not well-posed, it is *ill-posed*. It can be well-posed but be *ill-conditioned* (small changes in initial conditional have much bigger changes in solutions).

If your PDE models something physical, well the physical world will do something, usually the same something in similar conditions. Initial conditions are measured, so noise appears, which should not radically change what happens. Hence the three conditions given.

Of coures, the butterfly effect, where the butterfly's motion in some far away land, changes the intensity of a nearby storm is serious ill-conditioning.

An ill-posed problem, the backwards heat equation

Let $u(x,t)$ be a solution to $u_{xx} = u_t$, $u(x,0) = 0$ and look at $u(x,-1)$, this is not continuous with respect to the initial condition.

Alternately, look at forward solutions ($t > 0$) to $u_{xx} = -u_t$ with $u(x,0) = 0$ and look at $u(x,1)$

A famous ill-posed problem is the backwards heat equation. Consider it in one space dimension. Suppose at time $t = 0$, the tempature is everywhere zero. Can we say what was the tempature at time $t = -1$?

One can transform the equation, so we would be going forward in time. So driving in reverse is not the problem.

The forward heat equation is well posed.

An ill-posed problem

$$u(x,t) = \varepsilon \sin(nx) \exp(-n^2 t)$$

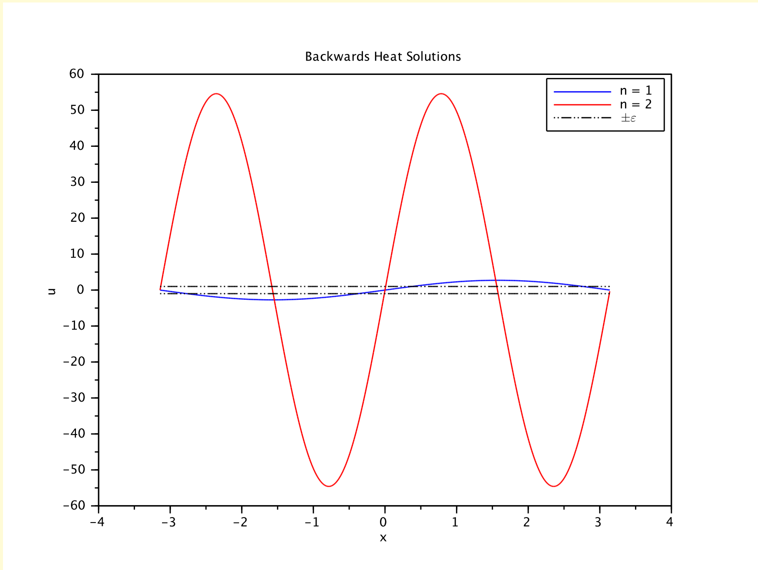
solves $u_{xx} = u_t$ and has $\sup |u(x,0)| \le \varepsilon$. But for $t = -1$

$$|u(x,-1)| = \varepsilon |\sin(nx)| \exp(n^2) \sim \varepsilon \exp(n^2) \to \infty$$

u_x changes $\sin(nx)$ to $n\cos(nx)$ and u_{xx} changes this to $-n^2 \sin(nx)$. The u_t changes $\exp(-n^2 t)$ to $-n^2 \exp(-n^2 t)$. So this u satisfies the heat equation $u_x x = u_t$.

The absolute value of u at time $t = -1$ is less than $\varepsilon \exp(n^2)$ which blows up as $n \to \infty$.

An ill-posed problem



Here is a plot of what is going on. Both the blue and red curves is end inside of our ε neighborhood in one unit of time. HIgher frequencies mean faster cooling.

I remember talking about the ill posedness of the backwards heat equation with the Dean at Harvey Mudd College who hired me as an adjunct in 1973. Likely it got me my first real job as a Mathematican that I had to interview for.

Picture sources

Hadamard picture is from Wikimedia commons
`https://commons.wikimedia.org/wiki/File:Hadamard2.jpg`

Plot is by the author using Scilab.