The Infinitude of Primes

This welcome is brought to you
by an Euler summation,
a topological proof,
and a trigonometric one liner.

Steven F. Bellenot

Aug 21, 2015

The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

There are Infinitely Many Primes

- Northshield 2015 a one liner
- Furstenberg 1955 (at age 20) a topological proof
- Euler 1737 (age 30) $\sum 1/p = \infty$

A recent one line trigonometric proof, a proof by a young Furstenberg and a classic result of Euler that proves more, that the sum of the reciprocals of the primes diverges.

We mention ages and dates as a aide for the students to idenitify with the researchers.

One Liner

Let $P = \prod_p p$ and suppose there are only finitely many primes so P is a finite integer.

$$0 < \prod_p \sin\left(\frac{\pi}{p}\right) = \prod_p \sin\left(\pi\frac{1+2P}{p}\right) = 0$$

Since $0 < \pi/p \leq \pi/2, 0 < \sin(\pi/p) \leq 1$, which shows the first inequility. Now $2P/p$ is an even integer, so we are adding a multiple of the period of $\sin x$, which shows the second equility. The number $1 + 2P$ is divisible by some prime q and for that factor $\sin((1 + 2P)/q)\pi) = 0$, which shows the last equility. The contradiction completes the proof.

In some sense, this is Euclid's proof with a twist. S. Northshield, A one-line proof of the infinitude of primes. Amer. Math. Monthly 122 (2015), no. 5, P 466.

Topology

- the union of open sets is open
- the complement of an open set is closed and the complement of a closed set is open
- a finite union of closed sets is closed.

We need to know open and closed sets are complements, that a finite union of closed sets is closed and any union of open sets is open.

Not everyone in my audience has taken a topology course, but all have seen open and closed sets. The proof is move about arithmetic sequences than topology.

Topological proof 1955

Define sets of the form $A_{j,n} = \{i : i = j \mod n\}$ to be open
Sets of the form $B_p = \{i : i = 0 \mod p\}$ are closed, it is $(\cup_{0<j<p} A_{j,p})^c$
$\{-1, 1\} = (\cup B_p)^c$ is open if there are only finitely many primes. But all non-trivial open sets are infinite.

$A_{j,n}$ is the collection of integers with remainder j when divided by n. B_p is the set of integers divisible by p, and it is closed being the complement of a union of open sets. Every integer, but ± 1 is in some B_p for some prime p. If there finitely many primes, then $\{-1, -1\}$ is open. But all non-empty open sets are infinite, which gives the contradiction.

There is more on this topic: https://en.wikipedia.org/wiki/Furstenberg%27s_proof_of_the_infinitude_of_primes

Euler and the Zeta Function $\zeta(1)$

$$
\begin{aligned}
&\left(1 + 1/2 + 1/2^2 + 1/2^3 + \cdots\right)\\
&\left(1 + 1/3 + 1/3^2 + 1/3^3 + \cdots\right)\\
&\left(1 + 1/5 + 1/5^2 + 1/5^3 + \cdots\right)\\
&\left(1 + 1/7 + 1/7^2 + 1/7^3 + \cdots\right)\\
&\vdots\\
&= 1/1 + 1/2 + \cdots + 1/84 + \cdots
\end{aligned}
$$

The zeta function $\zeta(s) = \sum_n 1/n^s$, so $\zeta(1) = \sum_n 1/n$. We are looking at a product of infinite series, one for each prime. Each line is geometric series with ratio $1/p$, which sums to

$$\frac{1}{1 - \frac{1}{p}}$$

. The equality claim is that the produces gives use the harmonic series $\sum 1/n$. The numbers in red when multiplied give the term in red. $84 = 2^2 \cdot 3 \cdot 7$. So unique factorization into primes provides the proof.

Punch line

If there are finitely many primes

$$\infty > \prod_p \frac{1}{1-\frac{1}{p}} = \prod_p (1 + \frac{1}{p} + \frac{1}{p^2} + \cdots) = \sum_n \frac{1}{n} = \infty$$

The last page showed the next to the last equality. The contradiction shows we have infinitely many primes.

Euler and the zeta function $\zeta(2)$

$$\begin{aligned}
&\left(1 + 1/2^2 + 1/2^4 + 1/2^6 + \cdots\right) \\
&\left(1 + 1/3^2 + 1/3^4 + 1/3^6 + \cdots\right) \\
&\left(1 + 1/5^2 + 1/5^4 + 1/5^6 + \cdots\right) \\
&\left(1 + 1/7^2 + 1/7^4 + 1/7^6 + \cdots\right) \\
&\vdots \\
&= 1/1 + 1/2^2 + \cdots + 1/84^2 + \cdots
\end{aligned}$$

The zeta function $\zeta(s) = \sum_n 1/n^s$, so $\zeta(2) = \sum_n 1/n^2$. A very similar product, this time the geometric series has ratio $1/p^2$ and product is $\sum_n 1/n^2$. Again $84 = 2^2 \cdot 3 \cdot 7$ is used as a typical case.

Punch line

$$\prod_p \frac{1}{1-\frac{1}{p^2}} = \prod_P (1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots) = \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$$

If there are finitely many primes the LHS is rational, but the RHS is transcendental (hence irrational).

If there are finitely many primes, the first product is rational. The number $\pi^2/6$ is transcendental.

Note π^2 is algebraic, if and only if, π is algebraic. The number π was shown to be transcendental by Lindemann (1882) long after Euler (1707-1783).

Appendix $\sum 1/p$ diverges

$$\begin{aligned}
-\ln(1-x) &= x + x^2/2 + x^3/3 + x^4/4 + \cdots \\
\prod(1/(1-1/p)) &= \sum 1/n = \infty \\
-\sum \ln(1-1/p) &= \sum(1/p + 1/2p^2 + 1/3p^3 + \cdots) \\
&= \sum(1/p) + \sum((1/p^2)(1/2 + 1/3p + \cdots)) \\
&< \sum(1/p) + \sum((1/p^2)(1 + 1/p + \cdots)) \\
&= \sum(1/p) + \sum((1/p^2)(1/(1-1/p)) \\
&= \sum(1/p) + \sum(1/(p(p-1)) \\
&= \sum(1/p) + C
\end{aligned}$$

Here p is always a prime. The first line is a Taylor series. you can check by taking derivatives. The third line, is from summing first lines with $x = 1/p$, p a prime. The second line imples that this sum is infinite. We pull out the $1/p$ out of each series. Factor a $1/p^2$ out of the remaining terms. These we increase to the geometric series $1 + 1/p + 1/p^2 + \cdots$. Replacing the series by its sum, we get the convergent

$$\sum_p \frac{1}{p(p-1)}.$$

The last equation gives

$$\infty = \sum_p \frac{1}{p} + C$$

So there are infinity many primes, and they are dense enough so the sum of the reciprocals diverges.

This is in the book, "Proofs from THE BOOK" where God keeps the most elegant proof of each mathematical theorem.