Generic, but ill conditioned

This welcome is brought to you
by invertible matrices,
matrices with n distinct eigenvalues,
and Wilkinson's Polynomial

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Generic properties and friends

- A Generic Property holds on an open dense set.
  - invertible $n \times n$ matrices
  - $n \times n$ matrices with n distinct eigenvalues
  - diagonalizable matrices?

Generic means typical, and in topology the points of an open dense set are typical. So we are going to show being invertable matrix is typical, having n distinct eigenvalues is typical. And what about diagonalizable matices?

A residual set, a countable insection of open dense sets, is a weaker notion is sometimes used for generic in topology. The irrationals are residual and dense but not open and dense.

The Set of Invertible $n \times n$ Matrices is Open

$$A^{-1} \text{ exists} \iff \det(A) \neq 0$$

The determinate is a polynomial on n^2 variables and so it is continuous. Thus the inverse image of $\{x \neq 0\}$ is open.

The inverse image of an open set by a continous function is open. The determination is continous since it is a polynomial. So the invertible matrices form an open set.

The Set of Invertible $n \times n$ Matrices is Dense

Suppose $\det A = 0$ and consider $p(t) = \det(A + tI)$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$
$$\det(A+tI) = \det(A) + t\left(\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right)$$
$$+ t^2(a_{33} + a_{22} + a_{11}) + t^3$$

Near zero, $p(t) \sim t^k$ some $k, 1 \leq k \leq 3$ and $A + tI$ is invertible.

To show denseness, we show each singular matrix is near an invertible one. We do this by using the permutation definition of the determinate. In the 3×3 case, the determinate of $A + tI$ is a cubic. If $\det(A) = 0$, then the cubic is non-zero, for small $t \neq 0$.

The set of Matrices with n Distinct Eigenvalues is Dense

Each A is similar to an upper diagonal matrix U, $A = PUP^{-1}$

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

whose eigenvalues are $\{u_{ii}\}$, we can perturb these to make U' with distinct eigenvalues and $A' = PU'P^{-1}$ will also have distinct eigenvalues.

Each matrix A is similar to an upper trianglar matrix U. We can perturb U slightly to U' so the diagonal entries are distinct. We can use the similarity transformation to find A', which has n distinct eigenvalues.

The set of Matrices with n Distinct Eigenvalues is Open

The coefficents of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is a continuous functon of the entries of A.

Rouché's theorem implies that if p has distinct zero's then there is a $\delta > 0$ so if the coefficents of q are within δ of those in p, then q has distinct roots.

Furthermore if p roots are real, then so are the roots of q.

We look at the characteric polynomial of our matrix with n distinct eigenvalues. The polynomial has n distinct roots. Perturbing A slightly, only perturbs the polynomial slightly and we can use Rouché, to say the perturb polynomial has n distinct roots.

The statement about real roots is to setup the next example.

A polynomial with real distinct roots, is relatively far from any polynomial with a non-trivial complex conjugate pair of roots.

Wilkinson's Polynomial

$$w(x) = \prod_{i=1}^{20}(x-i) = (x-1)(x-2)\cdots(x-20)$$

$$\begin{aligned} w(x) = x^{20} &- 210x^{19} + 20615x^{18} - 1256850x^{17} + 53327946x^{16} \\ &- 1672280820x^{15} + 40171771630x^{14} - 756111184500x^{13} \\ &+ 11310276995381x^{12} - 135585182899530x^{11} \\ &+ 1307535010540395x^{10} - 10142299865511450x^{9} \\ &+ 63030812099294896x^{8} - 311333643161390640x^{7} \\ &+ 1206647803780373360x^{6} - 3599979517947607200x^{5} \\ &+ 8037811822645051776x^{4} - 12870931245150988800x^{3} \\ &+ 13803759753640704000x^{2} - 8752948036761600000x \\ &+ 2432902008176640000 \end{aligned}$$

Wilkinson's polynomial shows that very small changes a coefficient can bring radical changes to the roots of a polynomial. The polynomial $w(x)$ has 20 distinct roots $1, 2, \dots 20$.

new stuff

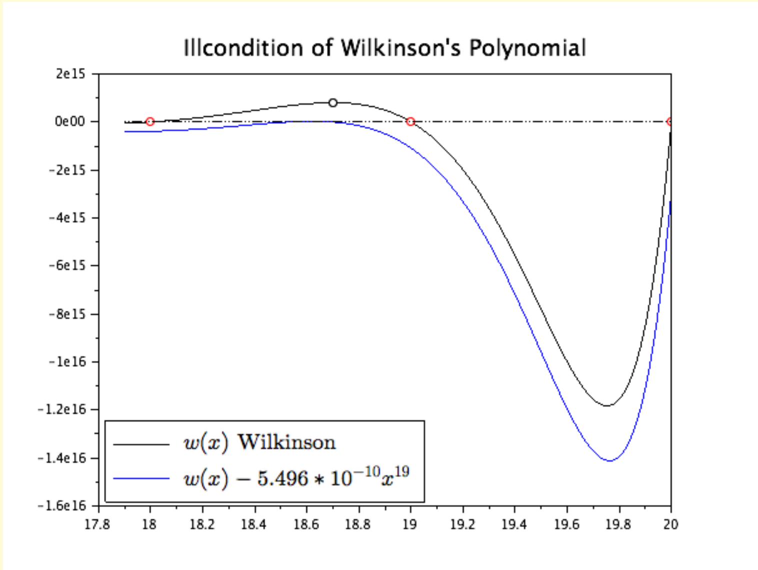
Ill Condition

Decrease the coefficent of x^{19} by a factor of $-210(2^{-31}) \sim -10^{-7}$ to -210.0000001192 and the roots become

1.0000	2.0000	3.0000	4.0000	5.0000
6.0000	6.9997	8.0073	8.9172	20.8469
$10.0953 \pm 0.6435i$	$11.7936 \pm 1.6523i$	$13.9924 \pm 2.5188i$	$16.7307 \pm 2.8126i$	$19.5024 \pm 1.9403i$

And yet a small change in the x^{19} coefficent produces a polynomial with complex roots far from the x-axis.

A Double Root Appears



Look at this plot of $w(x)$ between 18 and 19 (the red circles). There is a local maximum at the black circle which we will lower to make a double root. This doesn't take much, since x^{19} grows fast.

When I first heard this in a colloquium, I was surprised. Later, I discovered how easy it was to show the ill conditioning using only calculus 1.

The Set of Diagonalizable Matrices is not Open

$$A = \begin{bmatrix} 0 & \varepsilon \\ 0 & 0 \end{bmatrix}$$

is not diagonalizable.

This simple example shows that the set of diagonalizable matrices is not open. The zero matrix is diagonalizable, it is diagonal. But the matrix shown above has only one eigenvalue 0 (as a double root). But not with two independent eigenvectors. Its eigenspace is only 1 dimensional. It has a non-trival generalized eigenvector.

Picture sources

the graph was made by the author using Scilab.