Hyperboloid Model Students

This welcome is brought to you
by Poincaré,
Klein,
and Beltrami

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The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Escher Circle Limit IV

Poincaré Disk (Beltrami) lines are circles $\perp$ limit circle

Escher's Circle Limit IV is a tesselation of the Poincaré Disk. In this model of the hyperbolic plane, lines are circles that intersect the boundary limit circle perpendicularly. Beltrami found this model first. This is a conformal model, all angles are shown correctly in a conformal model.

The distance is not the Euclidean distance. The distance grows as one nears the limit circle. The figures in Escher's drawing all have the same size.

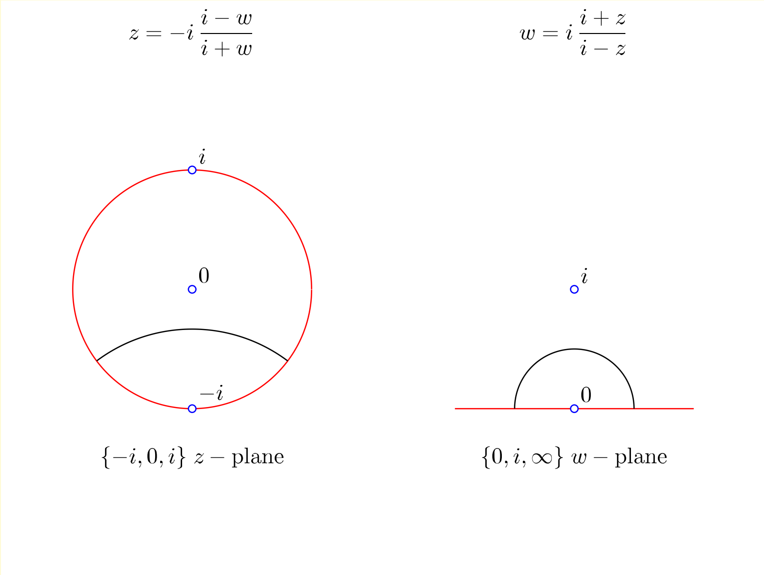
Half Plane Model

Poincaré half plane (Beltrami), lines are circles $\perp$ x-axis

In this model of the hyperbolic planes, lines are either rays from the x-axis, parallel to y-axis or circles which intersect the x-axis perpendicularly. Again named for Poincaré, but discovered by Beltrami. It too is conformal. The picture is the image of the Circle Limit IV via a Möbius transformation.

Again all the figures are the same size. The distances grow as one nears the x-axis.

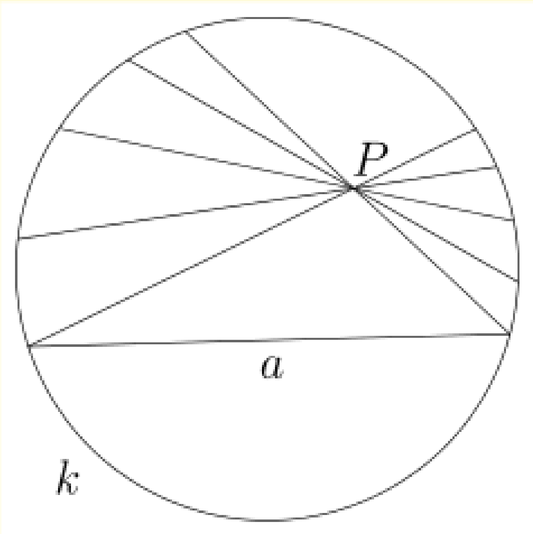
Möbius



This is the Möbius trasnsformation that connects these two models. The point $-i$ goes to 0, 0 goes to i and i goes to ∞. The red circle and red x-axis are limits outside the model. The black circluar arcs are mapped onto one another.

This shows these two models, the disk and the half plane are isometric models of hyperbolic space.

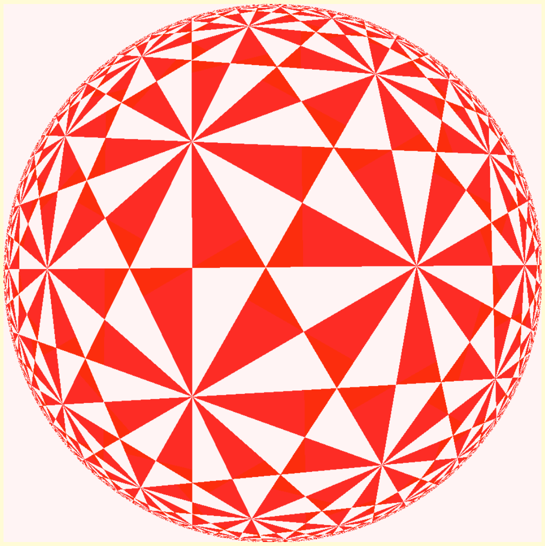
Klein (chord) Disc Model



Klein disk (Beltrami), lines are chords, angles distorted.

Note the many lines through P which are parallel to the line a. Since the end points of a chord are not part of the line, even the chords that intersect a at the limit circle, don't in the model. The chord model is not conformal, but chords perpendicular to a diameter in the Euclidean sense, are perpendicular in the model too. This model is named for Klein, who is one of my mathematical great great great grand fathers. Again Beltrami was the first to use the model.

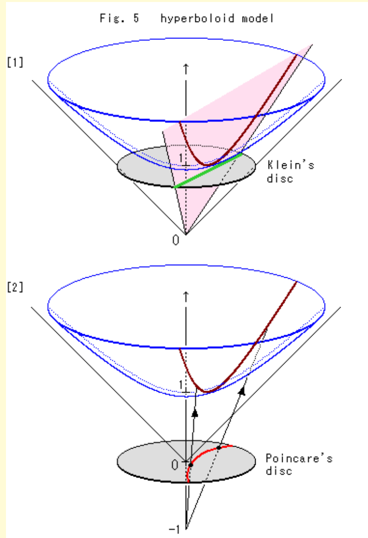
Hemisphere



Lines are semi-circlar chords, angles are distorted.

This is really the disk model, with the chords on the hemisphere. The lines became semi-circles. Not conformal.

Minkowski Hyperboloid Model



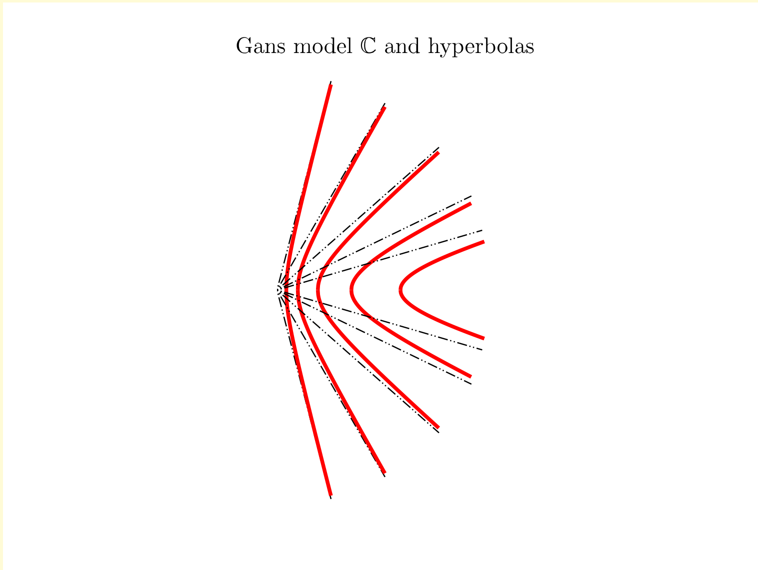
Minkowski or Lorentz (Weierstrass). Flattened Gans.

The Hyperboloid Model uses one sheet of a hyperboloid of two sheets. Here we use $x^2 + y^2 - z^2 = -1$. Lines can be thought of intersections of planes through the origin determined by chord's in Klein's disk at $z = 1$ (top). Or rays from $(0, 0, -1)$ through lines in Poincaré's disk at $z = 0$. This Model is named for Minkowski but was used by Weirstrass before. Lorentz space time. This provides a way of showing the Poincardísk and the Klein disk are isometric. This model has direct applications to special relativity.

With a rotation, we can assume that the pink plane has equation $z = ax$ with $a > 1$ and that the chord is perpendicular to the x-axis. This makes the intersection live on a surface, a cylinder over a hyperbola in the xy-plane given by $(a^2 - 1)x^2 - y^2 = 1$. We can parametrize the curve by letting $y = t$.

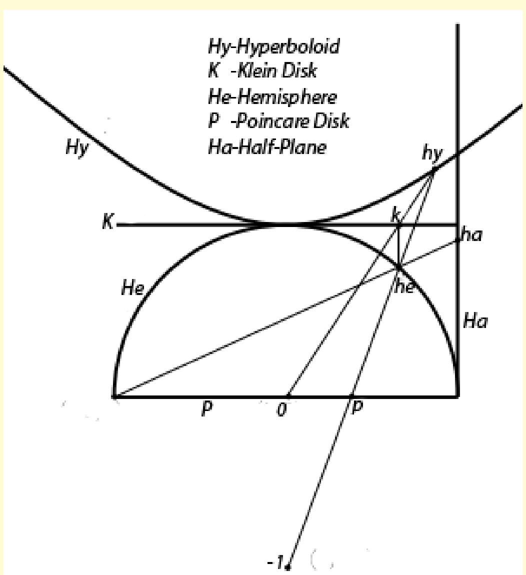
$$z = ax$$
$$z^2 = x^2 + y^2 + 1$$
$$a^2x^2 = x^2 + y^2 + 1$$
$$(a^2 - 1)x^2 - y^2 = 1 \qquad \text{hyperbolic cylinder}$$
$$y = t$$
$$(a^2 - 1)x^2 = 1 + t^2$$
$$\text{Parametric equations}x = \sqrt{1 + t^2}/\sqrt{a^2 - 1}$$
$$y = t$$
$$z = a\sqrt{1 + t^2}/\sqrt{a^2 - 1}$$

Gans – Flattened Hyperboloid



Gans's model projects the lines of the hyperboloid model onto two dimensions. Each hyperbolic line becomes a hyperbola. This model uses the whole plane.

Relationships



This picture provides linkage between all the models. The Klein model is at $z = 1$, the Poincaré Disk is at $z = 0$. The hemisphere is on the unit sphere, the half place is on the plane $x = 1$.

There is only one hyperbolic geometry in each dimension so any two models must be isometric.

Picture sources

Escher Circle Limit IV picture can be found at
`https://mcescher.com/gallery/mathematical/#iLightbox[gallery_image_1]/14`

Escher upper half plane is fronm
`http://arkadiusz-jadczyk.eu/blog/2017/04/24/`

Möbius map by the author using Scilab

Klein model picture is from
`https://commons.wikimedia.org/wiki/File:Klein_model.svg`

Hemisphere model could be from
`https://i.stack.imgur.com/2I1kPm.png`

Hyperboloid Model graphs are from
`http://web1.kcn.jp/hp28ah77/us3_poinc.htm`

Gans Model by the author using Scilab

5 Model Relations picture is from
`https://en.wikipedia.org/wiki/File:Relation5models.png`