Long time coming

This welcome is brought to you
by walking randomly,
waiting forever, and
returning with probability one.

Steven F. Bellenot

Aug 23, 2019

The talk is a slide show. The slides are framed in yellow rectangles. The quotation that follows, is what might have been said while the audience was looking at the slide. The blue comments like this one were added later and not part of the welcome. The title frame above was not the original.

Random walk on $\mathbb{Z}$ starting at zero

If you are at state i, you flip a coin to decide if you move up or down.

- With probability one, returns to zero infinitely often.
- But the expected return time is never (infinity).
- It is like waiting for Godot, a play that suggests life is full of suffering (written in French)

This is the classic Drunkard's Random walk in one dimension. The drunk always returns to zero, with probability one. Yet the expected wait time is infinity. Spoiler alert, in the play Godot never comes.

Expect me at the same time

Let x_i be the expected time (number of steps) starting at i to reach 0. From i, it is equally likely to go to $i \pm 1$ so

$$x_i = \frac{1}{2}(1 + x_{i-1}) + \frac{1}{2}(1 + x_{i+1})$$

$$-\frac{1}{2}x_{i-1} + x_i - \frac{1}{2}x_{i+1} = 1$$

Two special cases, by symmetry $x_1 = x_{-1}$ which gives $x_0 - x_1 = 1$ And for $i = 1$ we use 0 instead of x_0 since we have arrived! $x_1 - \frac{1}{2}x_2 = 1$

The expected time (average number of steps) equation is obtained by adding the possible outcomes. Half the time we go to $i - 1$ and then the expected time is $1 + x_{i-1}$ steps. The 1 is for the step we just took. The other half of the time is similar. We re-write into a standard linear equation, one for each i. So infinitely many equations in infinitely many unknowns.

Two special cases, by symmetry, $x_{-1} = x_1$ so $i = 0$ equation can be replaced with $x_0 - x_1 = 0$. This eliminates the equations for negative i. And second, we not use the x_0 amount from $i = 1$ since we are already at 0. Lets put this in matrix notation.

Solve Me

$$\left[\begin{array}{cccccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

This is the system we want to solve. We use row operations to make it upper triangular. The first one is replacing row 3 with row 3 plus 1/2 row 2 and then multiplying row 3 by 4/3. We get the matrix on the next slide. We continue by induction.

The next matrix has row 2 and row 3, then the result of $row3 + 1/2\,\text{row}\,2$ and finaly the new row 3 multiplied by 4/3.

$$\begin{array}{r|ccccc|c} r_2 & 0 & 1 & -\frac{1}{2} & 0 & \dots & 1 \\ r_3 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 1 \\ r_3 + r_2/2 & 0 & 0 & \frac{3}{4} & -\frac{1}{2} & \dots & \frac{3}{2} \\ 4r_3/3 & 0 & 0 & 1 & -\frac{2}{3} & \dots & 2 \end{array}$$

Continuing by induction. The next matrix has new row n and old row$(n+1)$, then the result of row$(n+1) + 1/2\,\text{row}\,n$, and finaly the new row$(n+1)$ multiplied by $2n/(n+1)$.

$$\begin{array}{r|cccccc|c} r_n & \dots & 0 & 1 & -\frac{n-1}{n} & 0 & \dots & n-1 \\ r_{n+1} & \dots & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 1 \\ r_{n+1} + r_n/2 & \dots & 0 & 0 & \frac{n+1}{2n} & -\frac{1}{2} & \dots & \frac{n-1}{2} \\ (2n)r_{n+1}/(n+1) & \dots & 0 & 0 & 1 & -\frac{n}{n+1} & \dots & n \end{array}$$

Upper Triangular Me

$$\left[\begin{array}{cccccccc|c} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & 0 & 0 & 0 & \dots & 2 \\ 0 & 0 & 0 & 1 & -\frac{3}{4} & 0 & 0 & \dots & 3 \\ 0 & 0 & 0 & 0 & 1 & -\frac{4}{5} & 0 & \dots & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{5}{6} & \dots & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 6 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{array}\right]$$

Now we use back substiution, to obtain equations for x_0 in terms of each x_i.

To infinity (and beyond?)

$$
\begin{aligned}
x_0 &= 1 + x_1 \\
x_0 &= 1 + (1 + \frac{1}{2}x_2) = 2 + \frac{1}{2}x_2 \\
x_0 &= 2 + \frac{1}{2}(2 + \frac{2}{3}x_3) = 3 + \frac{1}{3}x_3 \\
x_0 &= 3 + \frac{1}{3}(3 + \frac{3}{4}x_4) = 4 + \frac{1}{4}x_4 \\
x_0 &= 4 + \frac{1}{4}(4 + \frac{4}{5}x_5) = 5 + \frac{1}{5}x_5 \\
x_0 &= 5 + \frac{1}{5}(5 + \frac{5}{6}x_6) = 6 + \frac{1}{6}x_6
\end{aligned}
$$

Since $x_i \geq 0$, we have $x_0 \geq i$, which makes it ∞.

Clearly the pattern is $x_0 = n + \frac{1}{n}x_n$, so all of the x_i's are ∞.

A Formal Power Series

Let p_n be probability the walk is at 0 at the n-th step. Since we start at 0, $p_0 = 1$.

Let f_n be the probability that the first return to 0 is at n (The first passage time). The word return implies $f_0 = 0$.

$$P(x) = \sum_{n=0}^{\infty} p_n x^n, F(x) = \sum_{n=1}^{\infty} f_n x^n$$

$$P(x) = 1 + F(x)P(x) \text{ formally } P(x) = \frac{1}{1 - F(x)}$$

$F(1) \leq 1$, If $F(1) < 1$, then $P(1) < \infty$ and if $P(1) = \infty$ then $F(1) = 1$.

Next we want to show that a random walk returns with probability one. Let f_n be the probability the walk returns to 0 for the first time at step n. We want to show $F(1) = \sum f_n = 1$. To show this we shown it is equivalent to showing $P(1) = \sum p_n = \infty$, where p_n is the probability that walk is at 0 at step n. We create two formal power series, and obtain the relation

$$P(x) = \frac{1}{1 - F(x)}$$

by noting that after a first passage time, the walk "resets". Clear $F(1) \leq 1$ and if $F(1) < 1$, then $P(1) < \infty$, so $F(1) = \infty$ implies $F(1)$.

$$p_n = f_n + \sum_{i=1}^{n-1} f_i p_{n-i}$$

$$p_n = \sum_{i=1}^{n} f_i p_{n-i} \text{ Since } p_0 = 1$$

Either $j = n$ is first positive integer so that $X(j) = 0$, or there is $j < n$ which is the first positive integer with $X(j) = 0$. If $X(j) = 0$ then the probability that $X(n) = 0$ is p_{n-j}.

Tangent Line Approximation

$$\frac{\sqrt{n}}{\sqrt{n+1}} = \sqrt{1 - \frac{1}{n+1}} \doteq 1 - \frac{1}{2}\frac{1}{n+1}$$

$$\frac{n + 1/2}{n+1} = 1 - \frac{1}{2}\frac{1}{n+1}$$

The top line is the tangent line approximation for $f(x) = \sqrt{x}$ at $x = 1$. The bottom line is just algebra.

$$\sqrt{1 - \Delta x} \doteq 1 - \frac{1}{2}\Delta x$$

$$\frac{\sqrt{n}}{\sqrt{n+1}} \doteq \frac{n + 1/2}{n+1}$$

Summing it up

$$p_{2n} = \binom{2n}{n} \frac{1}{2^n} \frac{1}{2^n}$$

(n steps to the left and n steps to the right in any order).

$$\binom{2n}{n} \sim \frac{4^n}{2\sqrt{n}}$$

Induction, $n = 1$, $\binom{2}{1} = 2 = \frac{4^1}{2\sqrt{1}}$

$$\binom{2(n+1)}{n+1} = \frac{(2n+2)!}{(n+1)!(n+1)!}$$
$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \binom{2n}{n} = \frac{2(2n+1)}{n+1} \binom{2n}{n}$$
$$= 4\frac{n+1/2}{n+1} \binom{2n}{n} \doteq 4\frac{\sqrt{n}}{\sqrt{n+1}} \binom{2n}{n} = \frac{4^{n+1}}{2\sqrt{n+1}}$$

p_{2n} is the sum any sequence of moves of length $2n$ with exactly n choices of $+1$ and each of the $2n$ choices has probability $1/2$. We show by induction the estimate on $\binom{2n}{n}$. The $n = 1$ is easy. For the induction step, we factor out the highest terms from the factorials, do some algebra, and use our tangent line approximation.

This is an elementary way to avoid using Stirling's Formula.

Summing it up

This $p_{2n} \sim 1/\sqrt{n}$ and $\sum p_n = \infty$ and the walk returns with probability one.

We know the p-series $\sum 1/n^{1/2}$ diverges so $\sum p_n = \infty$ and the walk returns with probability one.

There is a joke about bad pennies in there somewhere. This was deeper than most talks. Fortuantely there were a half dozen students in the audience who had just taken a random walk course that summer.