The Riemann Hypothesis

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1 2 3 4 5 6 7 8 9

Bernard Riemann's 1859 Paper

On the Number of Primes Less than a Given Magnitude

. . . it is very likely that all of the roots of . . .

One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigations.

RH and PNT

Riemann Hypothesis:
The non-trivial zeros of $\zeta(s)$ all have real part $\frac{1}{2}$

Prime Number Theorem:
$\pi(x)$ **(number of primes $\leq x$)** $\sim Li(x) = \int_0^x \frac{dt}{\log t}$

PNT is equivalent to the non-trivial zeros of $\zeta(s) = \sum \frac{1}{n^s}$ having real part < 1.

RH and the Error Term

RH is equivalent to the statement for all $\epsilon > 0$,

$$|\pi(x) - Li(x)| = O(x^{\frac{1}{2}+\epsilon})$$

Roughly n-th prime & $Li^{-1}(n)$ have $\frac{1}{2}$ same digits.
RH is equivalent to

$$|\pi(x) - Li(x)| \leq \frac{\sqrt{x}\log x}{8\pi}, \text{ for } x \geq 2657$$

The error term is at least $Li(x^{\frac{1}{2}}) \log\log\log x$

RH and Primes are Random I

- $Li(x)$ **suggests** $1/\log x$ **is a density like function. Namely the integer** n **has a** $1/\log n$ **chance of being prime.**

- **False?** n **even. But one can factor this in.**

- **False? Chebyshev's bias, there are more** $4k+3$ **primes that** $4k+1$ **primes.**

RH and Primes are Random II

- **Möbius function $\mu(n) = 0$ if n is not square-free and is otherwise $(-1)^k$ where k is the number of distinct prime factors of n, $\mu(1) = 1$, $M(n) = \sum_1^n \mu(i)$**

- **Think of $\mu(n)$ being a coin toss on square-free numbers. If the heads/tails are random sequence of N tosses then with probability one the number of heads $-$ number of tails grows slower than $N^{\frac{1}{2}+\epsilon}$.**

- **RH is equivalent to $M(n) = O(n^{\frac{1}{2}+\epsilon})$**

The key ideas

- $n = \prod p_i^{\alpha_i}$ **where** p_i **are prime.**
- $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$
- $\sum \frac{1}{n^s} = (1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \cdots)(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \cdots)$
  $(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \cdots)\cdots = \prod_p (1 - 1/p^s)^{-1}$
- **complex** s**;** $\Re(s) > 1$**; PNT** $\Re(s) \geq 1, s \neq 1$

1 2 3 4 5 6 7 8 9

What Riemann did in the 1859 paper

- First to consider $\zeta(s)$ as $\zeta(\sigma + it)$.
- Found contour integral representation for $\zeta(s)$ good for all s except for the pole at $s = 1$.
- Doing the contour two ways, yields the functional equation:

$$\xi(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\frac{s(s-1)}{2}\zeta(s) = \xi(1-s)$$

What Riemann did in the 1859 paper

- The entire function $\xi(s)$ is zero exactly at the non-trivial $\zeta(s)$-zeros.
- Claimed $\xi(s) = \xi(0) \prod_\rho (1 - s/\rho)$.
- Claims about the distribution of these zeros ρ
- And program to go from $\pi(x)$ to the zeros of $\xi(s)$

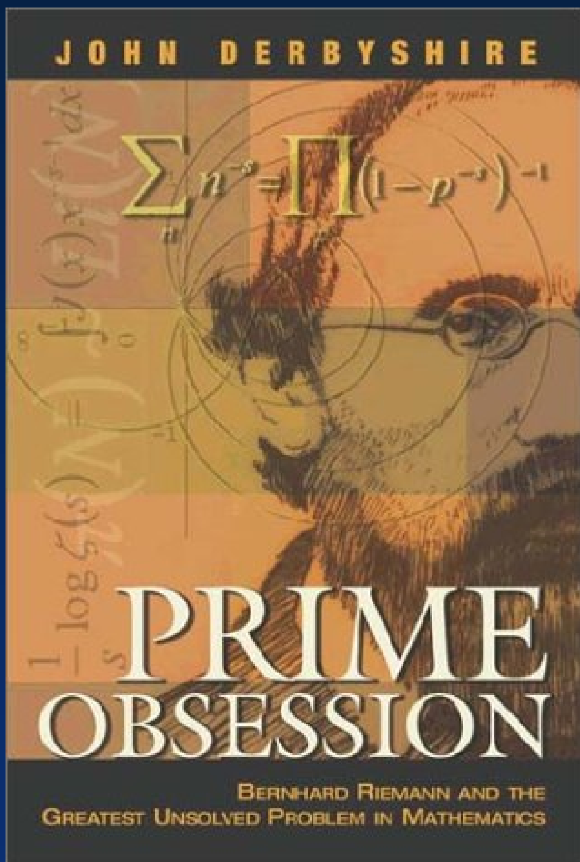
Best Partial Results

- All the non-trivial zeros are in the critical strip, $0 \leq \Re(s) \leq 1$ (Actually $0 < \Re(s) < 1$ by PNT.)
- At least 40.2% of these zeros lie on the critical line $\Re(s) = \frac{1}{2}$.
- Computers have checked billions of zeros.
- All but an infinitesimal proportion are ϵ close to the critical line.

$N(T)$, Counting Zeros in $0 \le \Im(s) \le T$

- $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$
- **There are at most $O(T)$ zeros with $\Re(s) > \frac{1}{2} + \epsilon$**
- $S(T) = N(T) - \frac{1}{\pi}\vartheta(T) - 1$ **is a measure of the error term. $S(T) = O(\log T)$, on average $S(T) = 0$, but $S(T)$ is unbounded.**
- **RH might be in trouble when $S(T) \approx 100$ which might be when $T \approx 10^{10^{10,000}}$**

April 2003


JOHN DERBYSHIRE
PRIME
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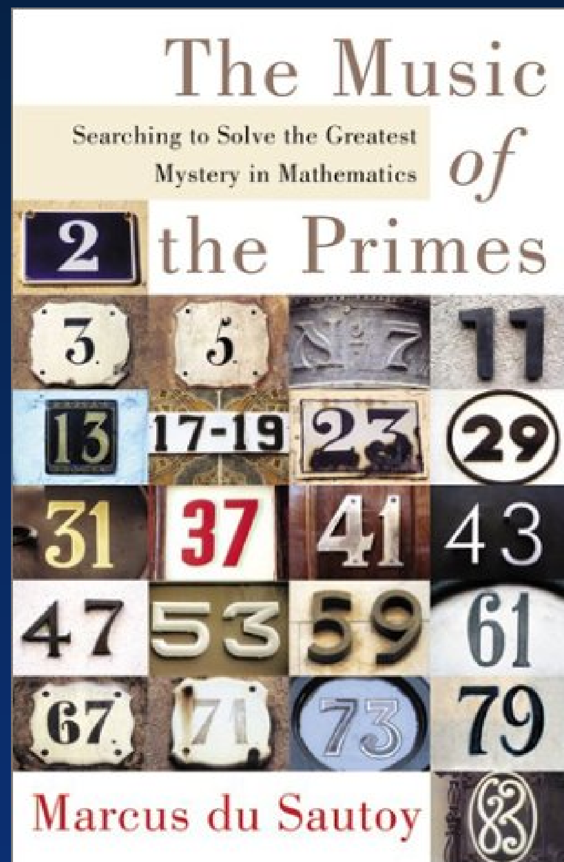
BERNHARD RIEMANN AND THE
GREATEST UNSOLVED PROBLEM IN MATHEMATICS


The Music of the Primes
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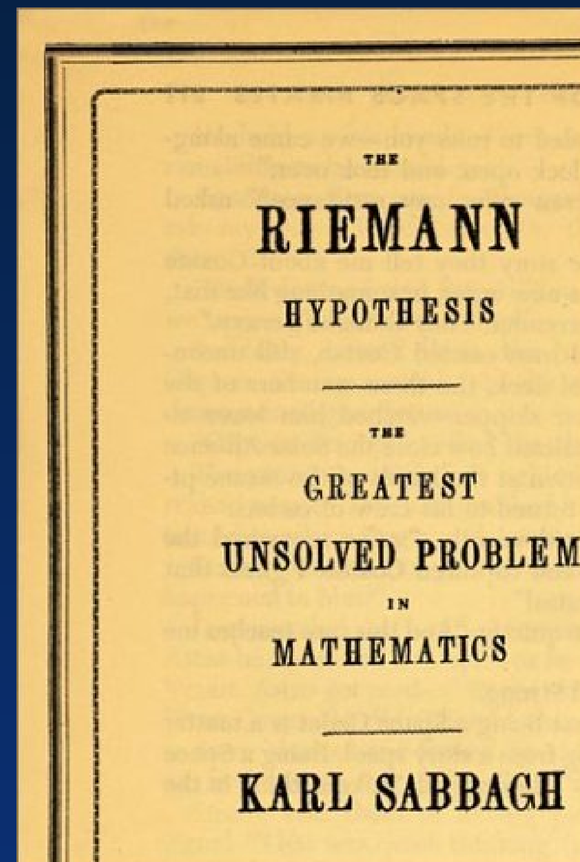
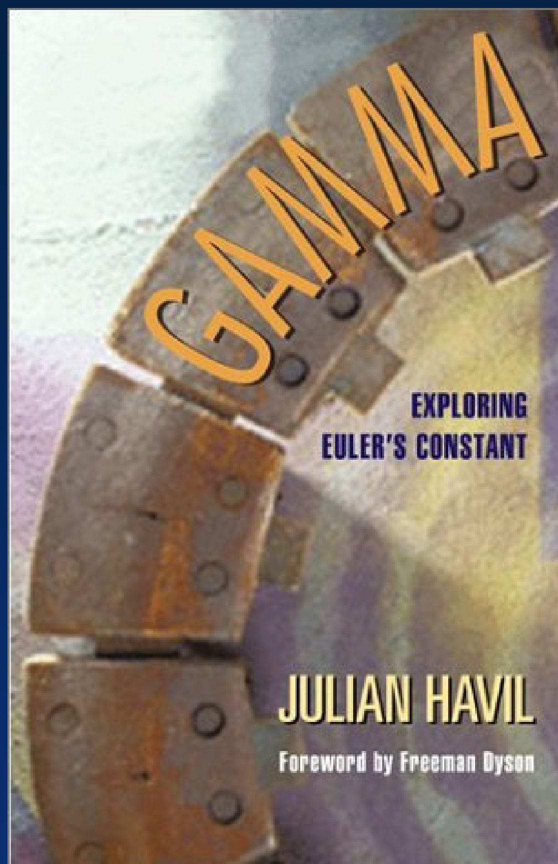
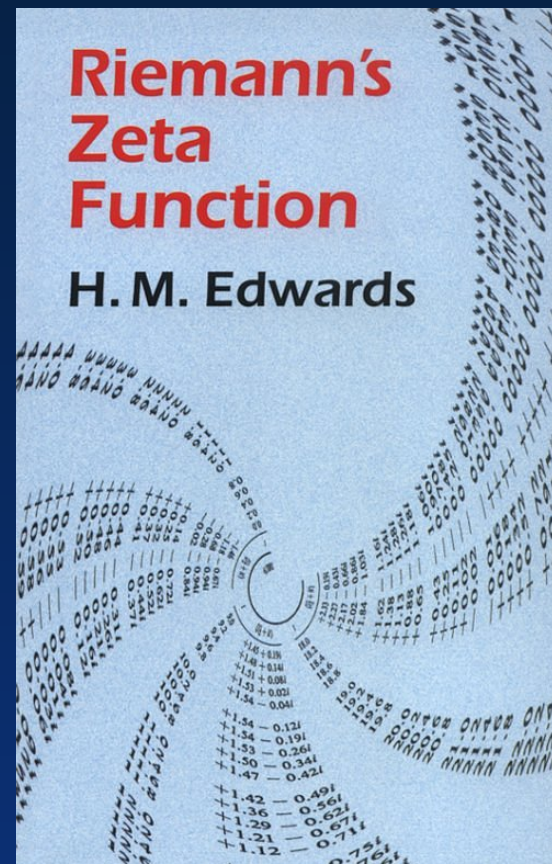

THE
RIEMANN
HYPOTHESIS
THE
GREATEST
UNSOLVED PROBLEM
IN
MATHEMATICS
KARL SABBAGH

$$\gamma = \lim_N \left(\sum_1^N \frac{1}{n} - \log N\right) \qquad \zeta(s) = \sum \frac{1}{n^s}$$



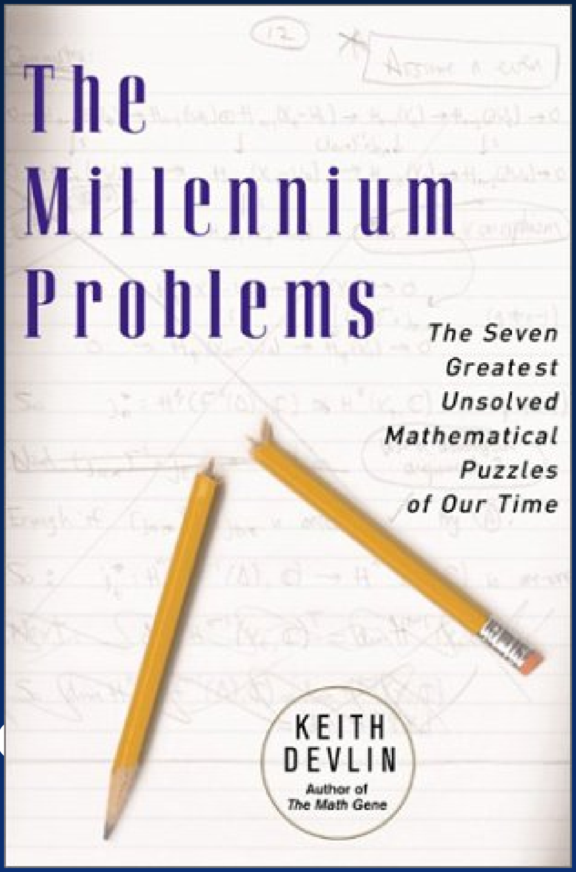


Clay Institute Millennium Problems

P versus NP
The Hodge Conjecture
The Poincaré Conjecture
The Riemann Hypothesis
Yang-Mills Existence and Mass Gap
Navier-Stokes Existence and Smoothness
The Birch and Swinnerton-Dyer Conjecture

Devlin's ordering and page count

44: Riemann Hypothesis
42: Yang-Mills Theory
26: P vs NP
26: Navier-Stokes Equations
32: Poincaré Conjecture
24: Birch & Swinnerton-Dyer Co
16: Hodge Conjecture



Erdős Quote

- On Mathematical prizes in general and about a $3,000 prize problem in particular
- "The prize money violated the minimum wage law"
- $1 million is roughly 96 years of $5 per hour work.
- $1 million is about 10 years work for a "average full professor in 2002: $96,380"

American Institute of Mathematics Conferences

1996 Seattle
PNT centennial
1998 Vienna
zeta function
2002 Courant
One of the founders owns Fry's Electronics

ζ ZetaGrid

Distributed computing project which uses 'spare' computer cycles to compute zeros of the zeta function. It uses a java program which anyone can download and run. In two years it has computed about a half a trillion zeros and verified RH for these.
http://www.zetagrid.net
Grid computing like SETI@Home

ζ ZetaGrid statistics Sep 8, 2003

- IBM Deutshland sponsored distributed grid computing using idle computer cycles. It involves over 8,000 workstations, yielding a peak power of over 4 TFLOPS and computes over a billion zeros a day.
- 2,891 people 8,053 computers 503 gigazeros 741 days
- SFB 22 million – number 1171 on the list

Computer Searches, Theory I

- **The functional equation $\xi(s) = \xi(1-s)$ and $\xi(\sigma)$ is real for real σ, so $\xi(\frac{1}{2}+it)$ is real.**
- $Z(t) = \xi(\frac{1}{2}+it) = exp(i\vartheta(t))\zeta(\frac{1}{2}+it)$ **is real valued.**
- $\zeta(\frac{1}{2}+it) = Z(t)\cos(\vartheta(t)) - iZ(t)sin(\vartheta(t))$
- **Gram points g_n where $\vartheta(g_n) = n\pi$. The function $\vartheta(t)$ is increasing and "easy" to compute.**

Computer Searches, Theory II

- ‘Expect’ $(Z(g_n))(-1)^n > 0$ “good” Gram points. When $Z(g_n)Z(g_{n+1}) < 0$, then $\zeta(s)$ has a zero between g_n and g_{n+1}.
- If $g_n + h_n$ are increasing and separate the zeros of $Z(t)$, and $h_N = 0$ then RH is true for $|t| < g_N$
- computing $Z(t)$ the Riemann-Siegel Formula $O(\sqrt{t})$-terms vs Euler-Maclaurin Summation $O(t)$-terms.

Are zeros good for anything?

- A counterexample?, but no million dollars.
- It found the pentitum floating point bug. ‘Near zeros’ have an application to twin primes.
- Statistics: not random – zeros have “repulsion”
- Quantum Mechanics: The energy levels of a heavy nuclei are so complex they are modeled on a statistical study of random matrices.

The Montgomery-Odlyzko Law

The distribution of the spacings between successive non-trivial zeros of the Riemann zeta function (suitably normalized) is statistically identical with the distribution of eigenvalue spacings of in a GUE (Gaussian Unitary Ensemble) operator.

$$1 - \frac{\sin \pi u}{\pi u}$$

Skewes number

- Gauss conjectured $\pi(x) \leq Li(x)$
- Littlewood (1914) showed it is false infinitely often
- Skewes (1933 with RH) the first failure $\leq e^{e^{e^{79}}} \approx 10^{10^{10^{34}}}$
- The estimate is now around to 10^{316} but might be 10^{178}. Estimates on total number of FLOPs in history is 10^{26}

Equivalence I (Robin 1984)

- $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$
- n **has** $(\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_k + 1)$ **divisors**
- $(1 + p_1 + p_1^2 + \ldots p_1^{\alpha_1})(1 + p_2 + \cdots + p_2^{\alpha_2}) \cdots (1 + p_k + \cdots + p_k^{\alpha_k}) = \sum_{d|n} d = \sigma(n)$

Equivalence I (Robin 1984)

- $H_n = \sum_1^n 1/j$ **the harmonic number.**
- **Euler's** $\gamma = \lim_n (H_n - \log n)$.
- $RH \iff \sigma(n) < e^\gamma n \log\log n$ **for all** $n \leq 5041$
- $\limsup \sigma(n)/n \log\log n = e^\gamma$
- $\sigma(n) < e^\gamma n \log\log n + 0.6482 \frac{n}{\log\log n}$ **for** $n \geq 3$

Equivalence II (Lagarias 2000)

- $H_n = \sum_1^n 1/j$ **the harmonic number.**
- $RH \iff \sigma(n) < H_n + \exp(H_n)\log(H_n)$ **for** $n \geq 2$
- RH **undecidable (Gödel sense) implies** RH **is true.**

Equivalence III (Beurling 1950?)

$N_{(0,1)}$ **is the set of functions of the form** $\sum_{k=1}^{n} c_k \rho(\theta_k/t)$ **where** $\sum_{k=1}^{n} c_k = 0$**,** $\theta_k \in (0,1)$ **and** $\rho(x) = x - \lfloor x \rfloor$**. TFAE**

- $\zeta(s)$ **has no zeros in** $\Re(s) > 1/p$
- $N_{(0,1)}$ **is dense in** L^p
- $\chi_{(0,1)}$ **is in the closure of** $N_{(0,1)}$ **in** L^p

History I, false claimers

- Stieltjes 1885
- Hardy – postcard to Bohr (Neils brother)
- Levinson 1974 – 98.4%, The last 1.6
- de Branges (Of Bieberbach fame)

History 2, nay sayers

- Long open conjectures in analysis tend to be false.
- it is arithmetic, to deep for analysis
- Landau
- Littlewood
- Turing

Riemann Program 1

- **Möbius function** $\mu(n) = 0$ **if** n **is not square-free and is otherwise** $(-1)^k$ **where** k **is the number of distinct prime factors of** n
- $J(x) = \sum_n \frac{1}{n}\pi(x^{1/n})$
- **Möbius inversion formula**
- $\pi(x) = \sum_n \frac{\mu(n)}{n} J(x^{1/n})$

Riemann Program 2

- **Derived from Euler's product formula (p prime)** $\zeta(s) = \sum_n n^{-s} = \prod_p (1 - 1/p^s)^{-1}$
- $\log \zeta(s) = s \int_0^\infty J(x) x^{-s-1}\, dx \quad \Re(s) > 1$
- **Inverse Fourier Transform**
- $J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} \quad a > 1$

Riemann Program 3

- Riemann found a contour integral definition of $\zeta(s)$ that was defined everywhere except for the pole at $s = 1$ namely

$$\frac{\Gamma(1-s)}{2\pi i}\int \frac{(-x)^s\;dx}{(e^x-1)x}$$

- Riemann found two ways to evaluate this contour integral yielding the functional equation. $\zeta(s) = \Gamma(1-s)(2\pi)^{s-1}2\sin(s\pi/2)\zeta(1-s)$

Riemann Program 4

- **Derived from the functional equation for** $\zeta(s)$
- $\xi(s) = \Gamma(1 + s/2)(s-1)\pi^{-s/2}\zeta(s)$
- **Satisfies the functional equation** $\xi(s) = \xi(1-s)$
- **And from a product formula**
- $\xi(s) = \xi(0)\prod_{\rho}(1 - \frac{s}{\rho})$ **where** ρ **is a zero of** ξ

Solving for $\log \zeta(s)$

$$\log \zeta(s) = \log \xi(0) + \sum_{\rho} \log(1 - s/\rho) \\ - \log \Gamma(1 + \frac{s}{2}) + \frac{s}{2} \log \pi - \log(s - 1)$$

Solving for $J(x)$

$$J(x) = Li(x) - \sum_{\rho} Li(x^{\rho}) + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} + \log \xi(0)$$

for $x > 1$, note that the last two terms are small for $x \geq 2$ the integral is 0.14001 and $\xi(0) = 1/2$ so $\log \xi(0) = -0.6931$

Solving for $\pi(x)$

Let N be large enough so that $x^{1/(N+1)} < 2$.

$$\pi(x) = \sum_{n=1}^{N} \frac{\mu(n)}{n} Li(x^{1/n}) + \sum_{n=1}^{N} \sum_{\rho} Li(x^{\rho/n}) + \text{lesser terms}$$

$$= Li(x) - Li(x^{1/2}) + \cdots + \sum_{\rho} Li(x^{\rho}) + \ldots$$

Filling the holes

- 1893 Hadamard – Zeros of entire functions
- 1895 von Mangoldt – Riemann's main formula (recast)
- 1896 Hadamard and de la Vallée Poussin – PNT independently
- 1905 von Mangoldt – estimate on number of zeros in strip

Hadamard: Zeros of entire functions

If $g(s)$ is an entire function and z_n is a list of non-zero zeros (with multiplicities) of $g(z)$ and $\sum 1/|z_n|$ diverges but $\sum 1/|z_n|^2$ converges, then $g(z) = z^m e^h(z) \prod (1 - z/z_n) e^{z/z_n}$. Futhermore the rate of growth of $g(z)$ limits the choices for $h(z)$.

Landau

- 1909 book, notation $\pi(x)$ and big oh.
- Pushed work to assistants in hospital via a ladder.
- Pushed work to assistants in train leaving on honeymoon.
- Fermat's last theorem.
- Bohr and Landau, number of roots $\frac{1}{2} + \epsilon \leq \Re(s)$ is infinitesimal

Landau

- Landau problems – including the twin prime conjecture.
- Traveled to England cause he didn't believe Littlewood existed
- Very rich, Jewish and lived in Nazi Germany