# Zeroing in on $\zeta$

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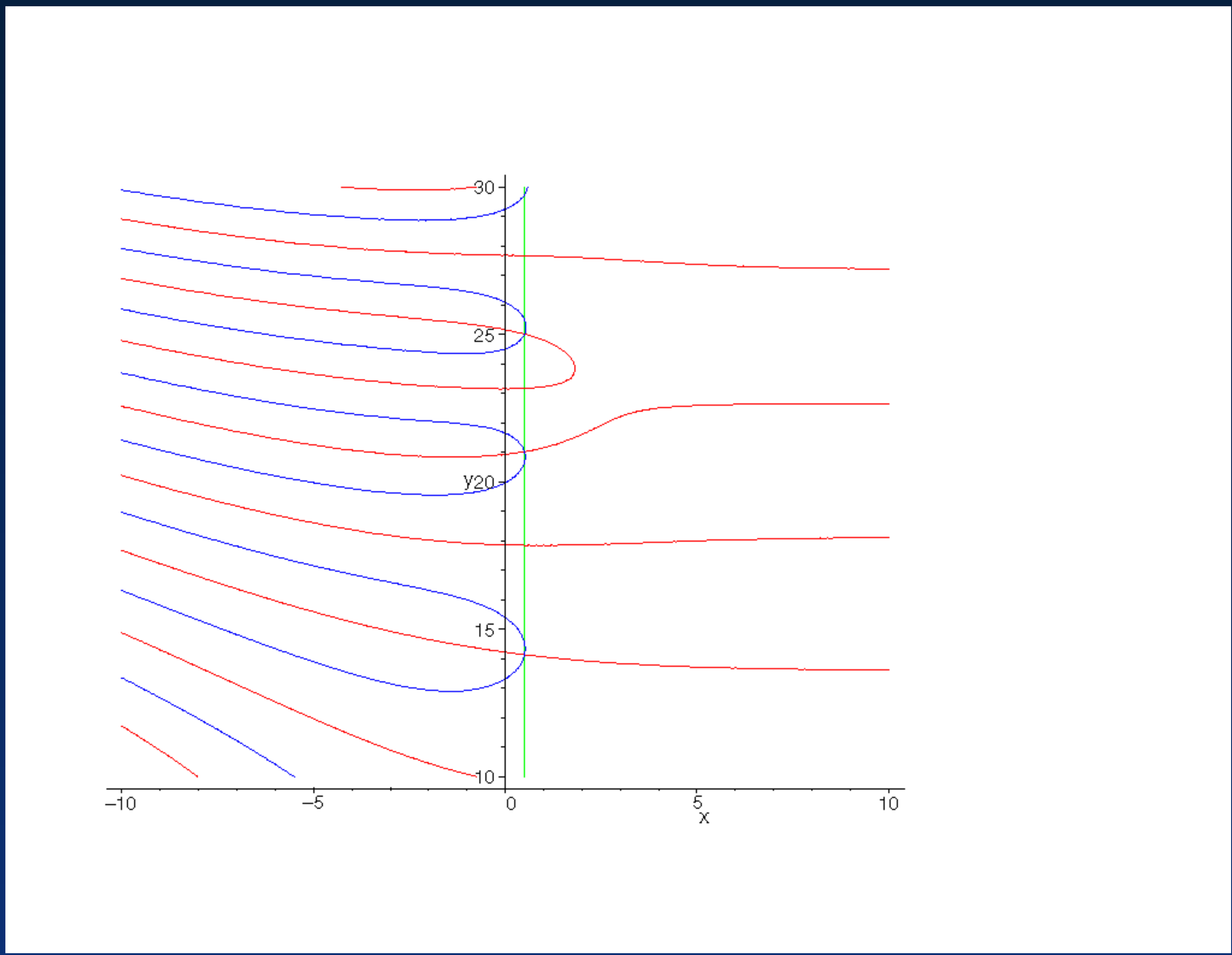
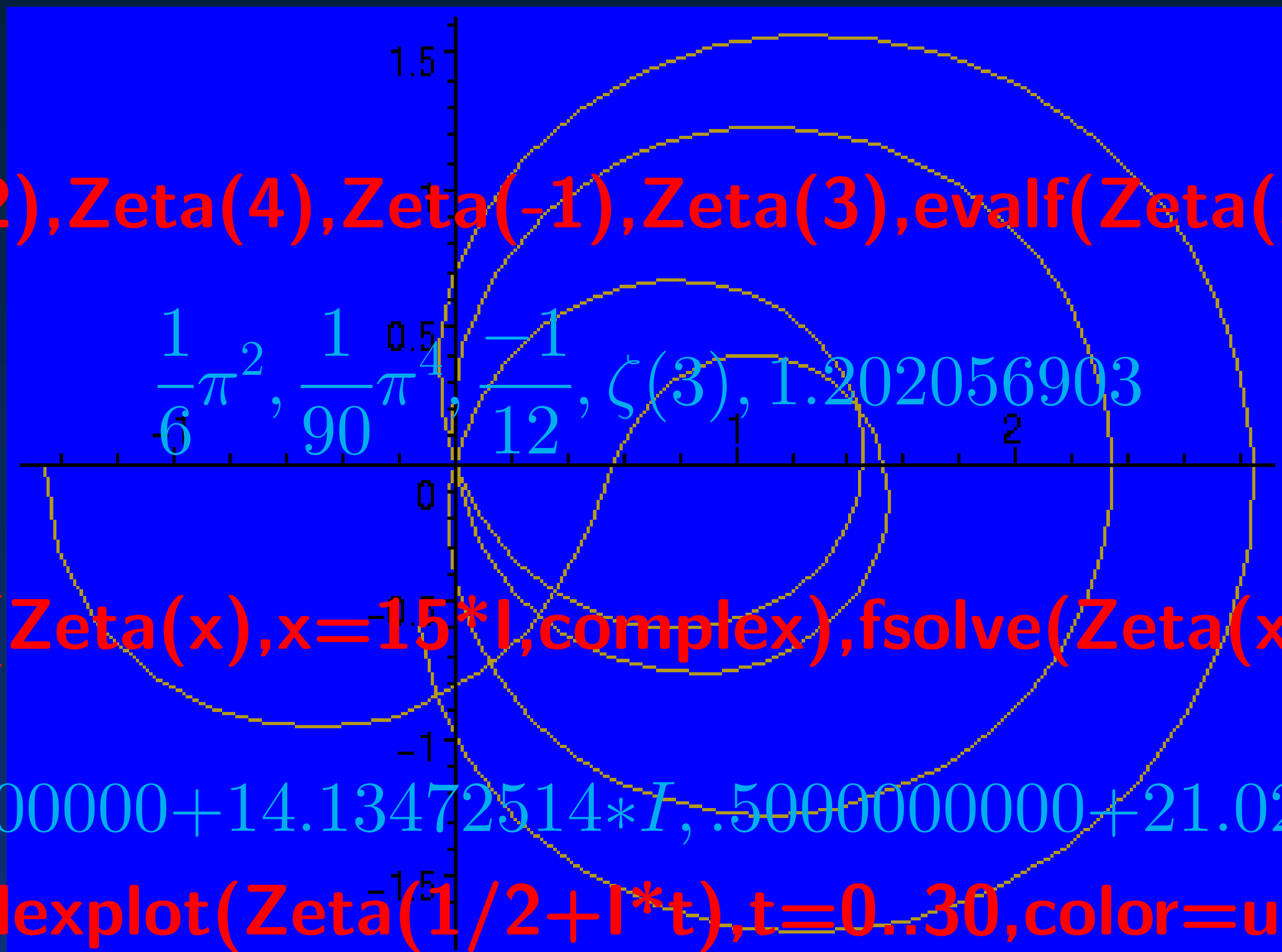
# $\zeta$ zeros and the Riemann Hypothesis



# Zeta is built into Maple




**Zeta(2),Zeta(4),Zeta(-1),Zeta(3),evalf(Zeta(3));**

$$\frac{1}{6}\pi^2, \frac{1}{90}\pi^4, \frac{-1}{12}, \zeta(3), 1.202056903$$

**fsolve(Zeta(x),x=15*I,complex),fsolve(Zeta(x),x=20*I,**

$$.5000000000+14.13472514*I, .5000000000+21.02203964*I$$

**complexplot(Zeta(1/2+I*t),t=0..30,color=ugly);**

# Euler, the master of us all

# The Basel Problem

$$\sum_n \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = ?$$

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It is almost

$$\sum_n \frac{1}{n(n+1)} = \sum_n \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots = 1$$

# $\zeta(2)$ converges

$$\frac{1}{n(n+1)} \le \frac{1}{n^2} \le \frac{1}{(n-1)n}$$

$$\sum \frac{1}{n(n+1)} \le \sum \frac{1}{n^2} \le 1 + \sum \frac{1}{n(n+1)}$$

$$1 \le \sum \frac{1}{n^2} \le 2$$

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$$\sum \frac{1}{n^2} = \sum \frac{1}{2^n n^2} + (\log 2)^2$$

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- $p(x)$ **has degree 2** $\qquad ax^2 + bx + c$
- **and** $p(2) = 0$ $\qquad (x-2)(px+q)$
- **and** $p(-1) = 0$ $\qquad k(x+1)(x-2)$
- **and** $p(0) = 6$ $\qquad -3(x-2)(x+1)$

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- $p(x) = 6(1 - \frac{x}{-1})(1 - \frac{x}{2})$

# Knowing nothing is almost everything

- $c_1, c_2, \ldots c_n$ **are the zeros of a polynomial** $p(x)$**, and** $x = 0$ **is not a zero, then**

- $p(x) = p(0)(1 - x/c_1)(1 - x/c_2) \cdots (1 - x/c_n)$.

- **If polynomial** $q(0) = 0$**, and** $0$ **is a root** $k$ **times, then** $p(x) = q(x)/x^k$ **is non-zero at** $0$ **and**

- $q(x) = x^k p(0)(1 - x/c_1)(1 - x/c_2) \cdots (1 - x/c_n)$

- $p(x) = p(0) \prod (1 - x/c_i)$.

# Lets make $\sin$ a honorary polynomial

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \text{zero is a single root}$$

$$\frac{\sin(\pi x)}{\pi x} = 0 \quad \text{for} \quad x = \pm 1, \pm 2, \pm 3 \ldots$$

$$\sin(\pi x) = \pi x \prod (1 - \frac{x^2}{n^2})$$

$$1 - \frac{x^2}{n^2} = (1 - \frac{x}{n})(1 - \frac{x}{-n})$$

# Series for the product

$$
\begin{aligned}
\prod &= (1 - \frac{x^2}{1^2})(1 - \frac{x^2}{2^2})(1 - \frac{x^2}{3^2})(1 - \frac{x^2}{4^2}) \cdots \\
&= 1 - x^2 (\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots) \\
&\quad + x^4 \Big( \frac{1}{1^2} (\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots) \\
&\qquad + \frac{1}{2^2} (\frac{1}{3^2} + \frac{1}{4^2} + \cdots) \\
&\qquad + \frac{1}{3^2} (\frac{1}{4^2} + \cdots) + \cdots \Big) + \cdots
\end{aligned}
$$

# Basel solved, coefficents of $x^3$

$$\sin \pi x = \pi x \prod (1 - \frac{x^2}{n^2})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin \pi x = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \cdots$$

$$-\frac{\pi^3}{3!} = -\pi \sum_n \frac{1}{n^2} \qquad \text{so} \qquad \sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$$

# Coefficents of $x^5$ and $\zeta(4)$

$$\frac{\pi^5}{5!} = \pi \sum_{n<m} \frac{1}{n^2 m^2}$$

$$(\sum \frac{1}{n^2})^2 = \sum \frac{1}{n^4} + 2 \sum_{n<m} \frac{1}{n^2 m^2}$$

$$(\frac{\pi^2}{6})^2 = \zeta(4) + 2\frac{\pi^4}{120}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

# $\zeta(2n)$

$$\zeta(6) = \frac{\pi^6}{945}$$

$$\zeta(8) = \frac{\pi^8}{9450}$$

$$\zeta(10) = \frac{\pi^{10}}{93555}$$

$$\zeta(12) = \frac{691\pi^{12}}{638512875}$$

$$\zeta(2n) = \frac{(-1)^{n-1}(2\pi)^{2n}B_{2n}}{2(2n)!}$$

$B_{2n} =$ Bernoulli number

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- Euler thought this was a wonderful technic and as good as any other solution. So did Riemann.

# Infinite products

- $\prod_{n=1}^{N}(1-a_n)$ **converges** $\iff$ **its** $\log$ **converges.**
- **The tangent approximation** $\log(1-x) \approx -x$.
- $\sum_{n=1}^{N} \log(1-a_n)$ **converges** $\iff$ $\sum_{n=1}^{N} a_n$ **converges**
- **So the RHS,** $\pi x \prod (1-\frac{x^2}{n^2})$ **converges since** $\sum \frac{x^2}{n^2}$ **converges. But does it converge to** $\sin(\pi x)$**?**

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- $\log \zeta(s)$ **can be similarly written as function of its zeros.**

# Zero hocus pocus I

$\sin x$ **has zeros at** $n\pi$**,** $\sin(x - \pi/2)$ **has zeros at** $n\pi/2$ **for odd** $n$**, so** $\sin x \cos x$ **has zeros at** $n\pi/2$**.**

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**But this is TRUE. It is the double angle formula** $\sin 2x = 2 \sin x \cos x$**.**

# Zero hocus pocus II

**Let** $g(x) = xe^{\gamma x} \prod e^{-x/n}(1 + x/n)$**. Note** $g(x)$ **has zeros at** $0, -1, -2, \cdots$ **and** $g(1) = 1$

**Note** $xg(x + 1)$ **also has the same set of zeros and the same value at** $x = 1$**, then**

$$xg(x + 1) = g(x)$$

**Since** $g(x)$ **is zero at** $0, -1, -2, \cdots$**, then** $g(1 - x)$ **is zero at** $1, 2, 3, \cdots$**, then** $g(x)g(1 - x) = k \sin(\pi x)$

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$\Gamma(x) = 1/g(x)$**, these formula translate to** $\Gamma(x+1) = x\Gamma(x)$ **so that** $\Gamma(n+1) = n!$ **and** $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$

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**Series $\sum (-1)^{n+1} n^{-x} = \eta(x)$ converges for $x > 0$, can be used to extend $\zeta(x)$.**

$1/(n+1)^x - 1/n^x$ **is about** $-x/n^{x+1}$

$$\eta(x) = (1 - 2^{1-x})\zeta(x)$$

$$\begin{aligned}
2\zeta(x)/2^x &= 2(1/2^x1^x + 1/2^x2^x + 1/2^x3^x + 1/2^x4^x + \cdots) \\
&= 2(1/2^x + 1/4^x + 1/6^x + 1/8^x + \cdots) \\
\eta(x) &= 1 - 1/2^x + 1/3^x - 1/4^x + 1/5^x - 1/6^x + \cdots \\
&= 1 + 1/2^x + 1/3^x + 1/4^x + 1/5^x + 1/6^x + \cdots \\
&\qquad - 2(1/2^x + 1/4^x + 1/6^x + \cdots) \\
\eta(x) &= (1 - 2^{1-x})\zeta(x) \\
\zeta(x) &= (1 - 2^{1-x})^{-1}\eta(x)
\end{aligned}$$

# Nonsense or Common Sense?

$$\eta(-1) = 1 - 2^1 + 3^1 - 4^1 + \cdots = 1/4$$

$$\eta(-2) = 1 - 2^2 + 3^2 - 4^2 + \cdots = 0$$

$$\eta(-3) = 1 - 2^3 + 3^3 - 4^3 + \cdots = -1/2$$

$$\eta(-4) = 1 - 2^4 + 3^4 - 4^4 + \cdots = 0$$

$$\eta(-5) = 1 - 2^5 + 3^5 - 4^5 + \cdots = 1/4$$

$$\eta(-6) = 1 - 2^6 + 3^6 - 4^6 + \cdots = 0$$

$$\eta(-7) = 1 - 2^7 + 3^7 - 4^7 + \cdots = -17/16$$

# Geometric Series

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \qquad |x| < 1$$

**This series explodes as $x \to 1$ ($x \to 1^-$) But $1/(1-x)$ is continuous at $x = -1$, hence**

$$\lim_{x \to -1^+} \frac{1}{1-x} = \frac{1}{1 - -1} = \frac{1}{2}$$

**A slight(?) abuse of notation yields**

$$\eta(0) = 1 - 1 + 1 - 1 + 1 \cdots = \frac{1}{2}$$

## Operate by $x\frac{d}{dx}$, limit as $x \to -1^+$

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \qquad |x| < 1$$

$$0 + 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2} \qquad |x| < 1$$

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2} \qquad |x| < 1$$

$$-1 + 2 - 3 + 4 - \cdots = \frac{-1}{(1 - -1)^2} = \frac{-1}{4}$$

$$\eta(-1) = 1 - 2 + 3 - 4 + \cdots = 1/4$$

# Operate by $x\frac{d}{dx}$, limit as $x \to -1^+$

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2} \qquad |x| < 1$$

$$1 + 2^2x^1 + 3^2x^2 + 4^2x^3 + \cdots = \frac{1+x}{(1-x)^3} \qquad |x| < 1$$

$$x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = \frac{x(1+x)}{(1-x)^3} \qquad |x| < 1$$

$$-\eta(-2) = -1 + 2^2 - 3^2 + 4^2 - \cdots = \frac{-1(1+-1)}{(1--1)^3} = 0$$

# More Terms

$$1^3x + 2^3x^2 + 3^3x^3 + \cdots = \frac{x(1+4x+x^2)}{(1-x)^4}$$

$$\eta(-3) = \frac{-2}{16} = \frac{-1}{8}$$

$$1^4x + 2^4x^2 + 3^4x^3 + \cdots = \frac{x(1+x)(1+10x+x^2)}{(1-x)^5}$$

$$\eta(-4) = 0$$

# See the Pattern?

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| $1-n$ | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ |
| $\zeta(n)$ | $\frac{\pi^2}{6}$ | ? | $\frac{\pi^4}{90}$ | ? | $\frac{\pi^6}{945}$ | ? | $\frac{\pi^8}{9450}$ |
| $1-2^{1-n}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | $\frac{7}{8}$ | $\frac{15}{16}$ | $\frac{31}{32}$ | $\frac{63}{64}$ | $\frac{127}{128}$ |
| $\eta(1-n)$ | $\frac{1}{4}$ | 0 | $\frac{-1}{8}$ | 0 | $\frac{1}{4}$ | 0 | $\frac{-17}{16}$ |
| $\frac{\eta(1-n)}{\eta(n)}$ | $\frac{3}{1}\cdot\frac{1!}{\pi^2}$ | 0 | $\frac{-15}{7}\cdot\frac{3!}{\pi^4}$ | 0 | $\frac{63}{31}\cdot\frac{5!}{\pi^6}$ | 0 | $\frac{-255}{127}\cdot\frac{7!}{\pi^8}$ |
| sign | $+1$ | 0 | $-1$ | 0 | $+1$ | 0 | $-1$ |
| $\cos\frac{\pi n}{2}$ | $+1$ | 0 | $-1$ | 0 | $+1$ | 0 | $-1$ |

$$90 = 15\cdot 3!, \qquad 945 = 63\cdot 3\cdot 5, \qquad 255 = 3\cdot 5\cdot 17,$$

$$9450 = (3\cdot 5)(3\cdot 5\cdot 6\cdot 7)$$

# Final Equations

$$\zeta(x) = \sum_n \frac{1}{n^x} \qquad x > 1$$

$$\zeta(x) = (1 - 2^{1-x})^{-1}\eta(x) \qquad x > 0, x \neq 1$$

$$\zeta(x) = (1 - 2^{1-x})^{-1} \lim_{t \to -1^+} \sum_n \frac{t^n}{n^x} \qquad x \neq 1$$

$$\zeta(1 - x) = \zeta(x)\, \pi^{-x}\, 2^{1-x}\, x!\, \cos\frac{\pi x}{2}$$

$$\sum_n \frac{1}{n^s} = \prod_p (1 - \frac{1}{p^s})^{-1} \qquad \text{Golden Key}$$