Color My World: Who let the computers in?

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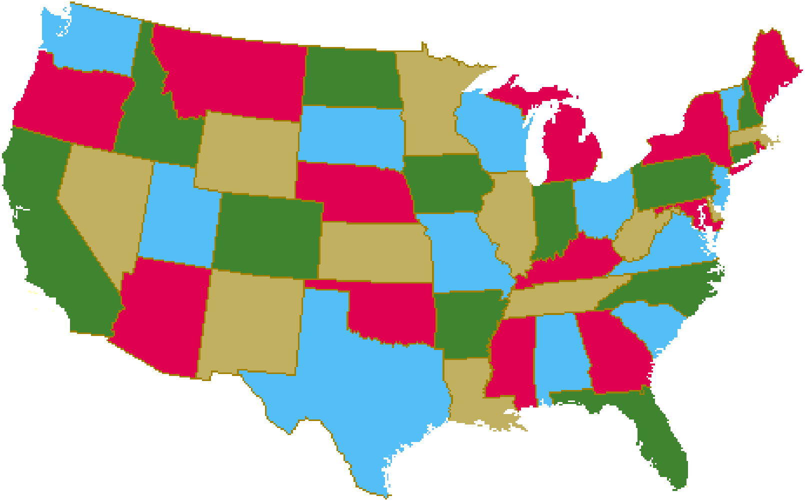
Outline

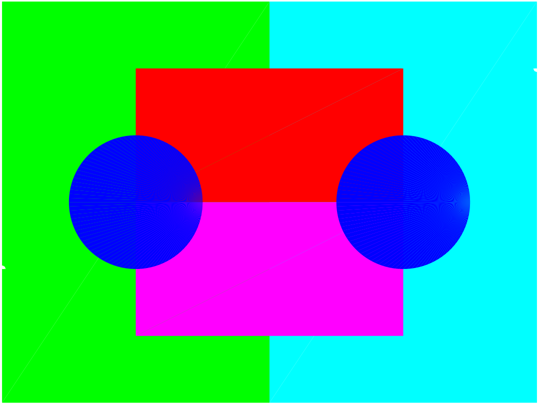
Four Color Conjecture

- Map statement, disproof?, Graph statement
- Early proofs?, Kempe chains,
- Discharging
- Reducible configurations
- Enter the computer
- Reaction and refinements
- Still open: Coloring the plane – point by point – Hadwiger Nelson

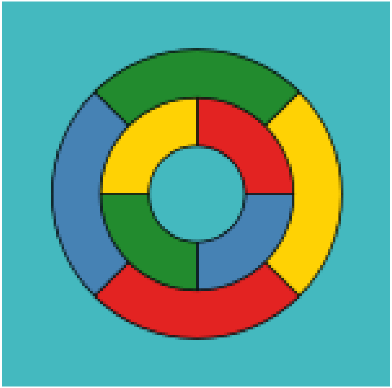
Map statement

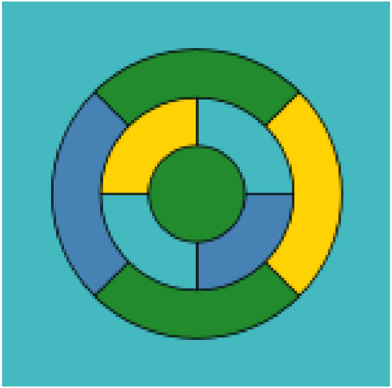


Not useful to Cartographers

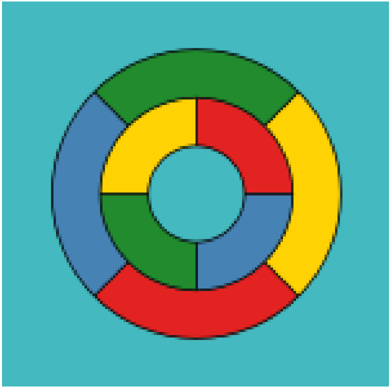


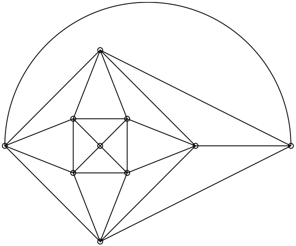
Disproof?





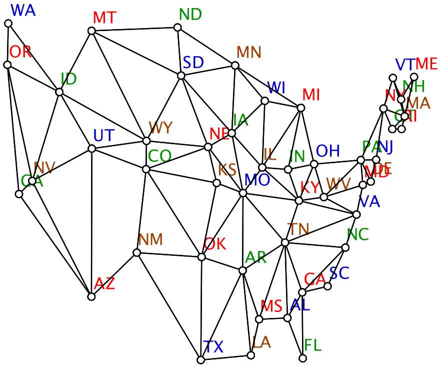
Dual





Graph statement

USA as a connected graph, states located at capitals



Colorful History

1852 open 124 years (Oct 23, 1852) Frank Guthrie.

1879 "Proved" by Kempe (a barrister).

1880 "Proved" by Tait (a mathematical physicist).

1890 Heawood found Kempe's flaw, proved that five colors suffice.

1891 Petersen found Tait's flaw.

1976 Computer proof – Appel and Haken.

1995 Better Computer proof – Robertson, Sanders, Seymour and Thomas.

2005 Coq proof assistant – Gonthiers – 3 CPU days.

A long and winding room

1904 Discharging – Wernicke

1913 Reducible – Birkhoff

1922 $|V| \leq 25$ can be 4 colored – Franklin

1926 $|V| \leq 27$ can be 4 colored – Reynolds

1940 $|V| \leq 36$ can be 4 colored – Winn

1967-71 Algorithms for Computer Proof – Heesch

1970 $|V| \leq 39$ can be 4 colored – Ore and Stempe

1976 $|V| \leq 95$ can be 4 colored – Mayer

The proof template

Suppose there is a counterexample, then there is a counterexample with the fewest vertices. And among those with the fewest vertices there is one with the most edges. All regions are triangles.

There is a finite collection of unavoidable configurations each of which can be reduced to a smaller counterexample.

An unavoidable set of reducible configurations.

Planar Graphs have a vertex of Degree 5 or less

Triangle faces, $2E = 3F$; Euler $V - E + F = 2$.

$$\begin{aligned} 3V - 3E + 3F &= 6 \\ 3V - 3E + 2E &= 6 \\ 3V - E &= 6 \\ 3V &= E + 6 \end{aligned}$$

If all vertices have degree 6 or more

$$2E = \sum deg(v) \geq 6V = 2E + 12$$

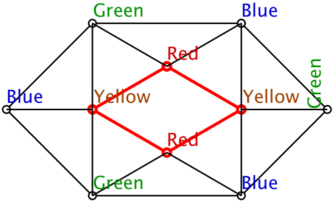
For later: $\sum(6 - deg(v)) = 6V - 2E = 12$

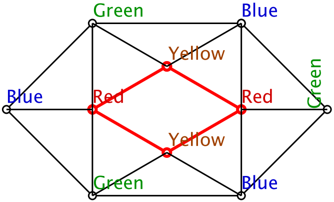
Kempe Chains

If a graph G is properly colored, a *Kempe Red/Yellow Chain*, K, is a maximal connected subgraph of vertices colored Red or Yellow.

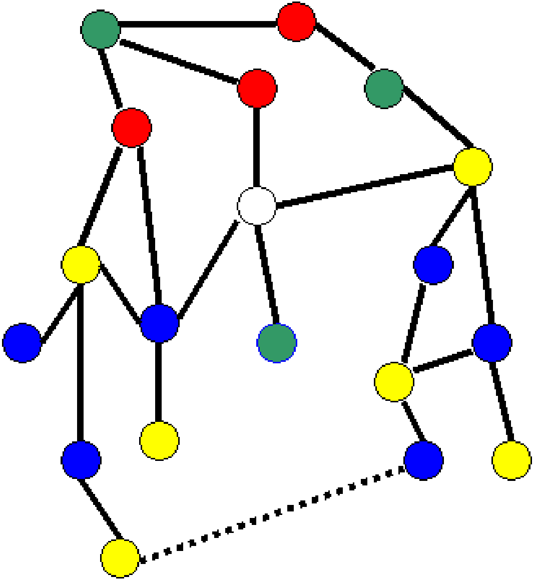
If you reverse the colors in a Kempe Red/Yellow Chain: Change the Reds to Yellow and the Yellows to Red, then G is still properly colored.

Kempe Red/Yellow Chain Swap

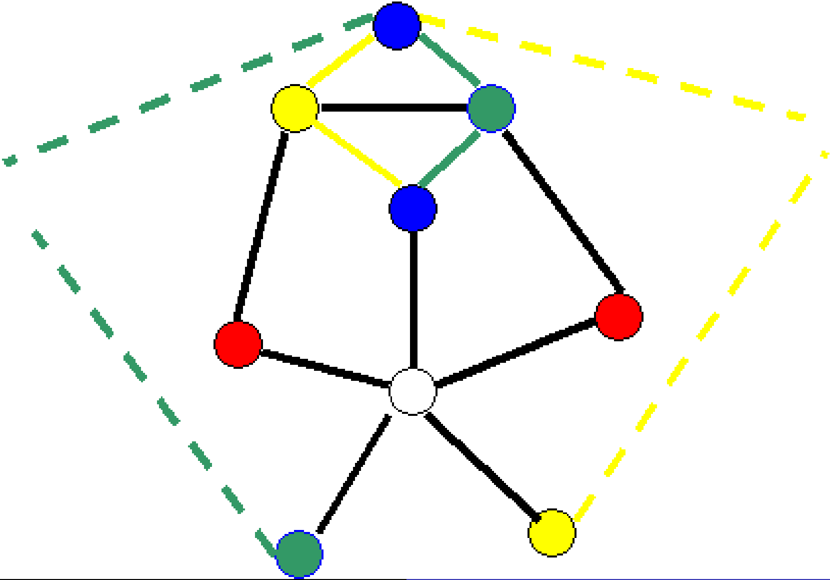




Degree 4 Case



Degree 5 Flaw



Definitions

- A *major* vertex has degree greater than 6, otherwise the vertex is minor.
- A *light* edge adjoins minor vertices.
- A k-vertex has degree k.
- A k^+-vertex has degree k or more.
- A k^--vertex has degree k or less.

Discharging example

In 1904, Wernicke introduced the discharging method to prove:

Theorem

If a planar triangulation has $\delta = 5$ then it has a light edge.

Charge the vertices with $6 - d(v)$. Euler implies the charges sum to 12.
Discharge rule: Each 5-vertex gives 1/5 charge to each neighbor.
There are still vertices of positive charge.
An 8^+ vertex still has negative charge.
A 7 vertex with positive charge must have six or more adjacent 5 vertices as part of its surrounding 7 cycle. Two must be adjacent.
If a 5-vertex or 6-vertex has a postive charge, it must have a 5-vertex neighbor and we have a light edge.

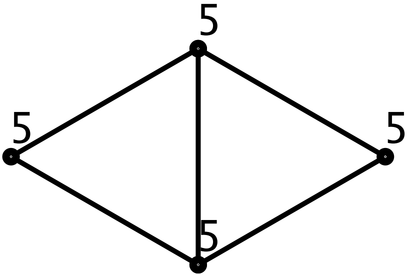
Second Proof

Discharge rule: Each 5-vertex gives $1/5$ charge to each major vertex neighbor.

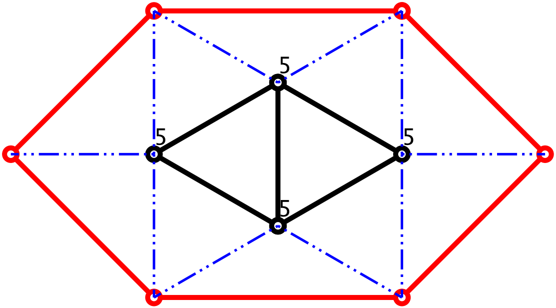
A 5-vertex with a positive charge has a minor vertex neighbor.

Otherwise there is a 7-vertex with positive charge and hence has six 5-vertices as neighbors.

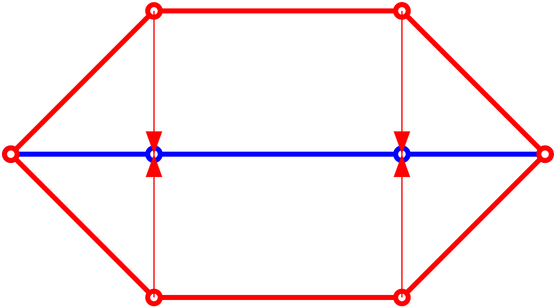
Reducible Configuration – Birkhoff 1913



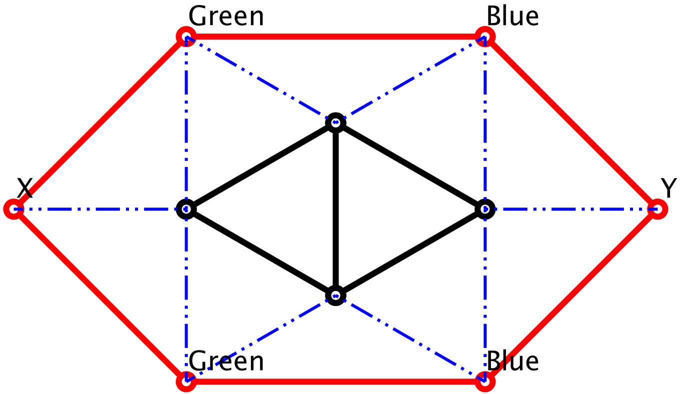
Birkhoff Diamond with Ring



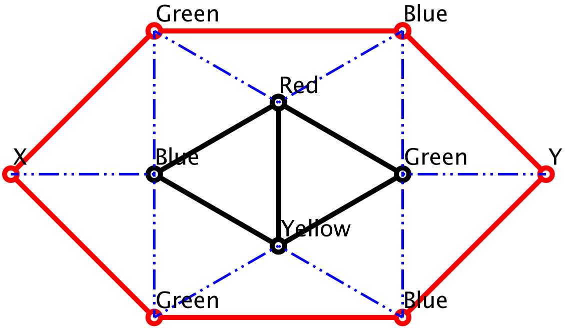
Induction Step



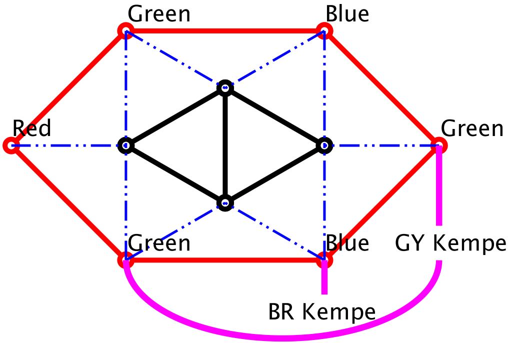
Nine Cases



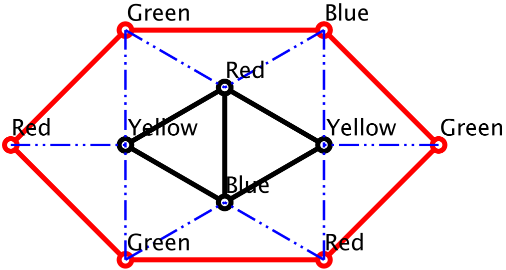
Four Easy Cases



Sample Hard Case – Kempe Chains



Sample Hard Case Solved



Heinrich Heesch

4CT could be proved by finding an unavoidable set of reducible configurations. In 1950 conjectured that the size of the set around 10,000.

Developed methods for a computer aided proof. Discharging in particular. Research funding was canceled. He could not get the computer time. (1967-1971)

The number of 4-colorings of a n-cycle

Color with $c_1, \ldots c_n$.
If $c_1 \neq c_{n-1}$, then $c_1 \ldots c_{n-1}$ colors $(n-1)$-cycle with 2 choices for c_n.
If $c_1 = c_{n-1}$, then $c_1 \neq c_{n-2}$ and $c_1 \ldots c_{n-2}$ color $(n-2)$-cycle with 3 choices for c_n

$$A(n) = 2A(n-1) + 3A(n-2)$$

$$A(n) = 3^n + 3(-1)^n$$

Number of colorings to check for reducibility

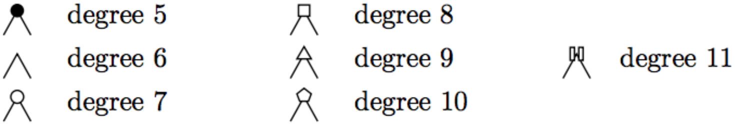
n	number of 4-colorings	canonical estimate
6	732	30.5
8	6564	273.5
10	59052	2460.5
12	531444	22143.5
14	4782972	199290.5

Canonical if $c_1 =$ Blue, $c_2 =$ Green, next new color is Red. roughly $4 \cdot 3 \cdot 2 = 24$.

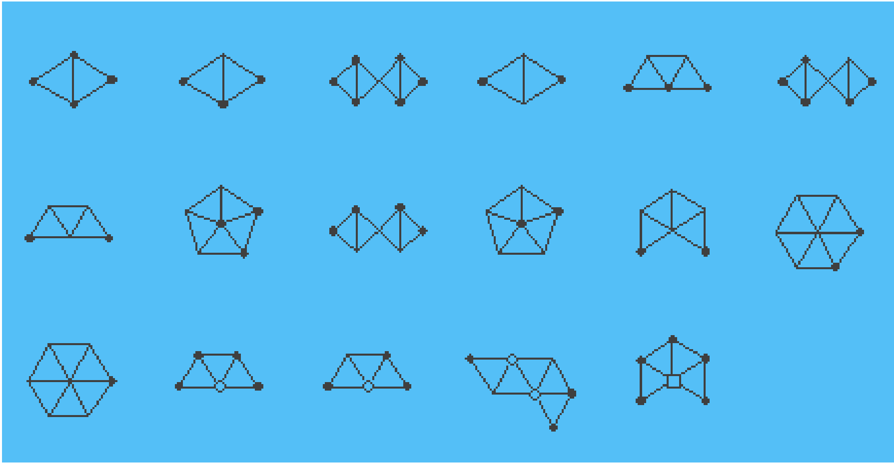
The 90 minute threshold

Appel and Haken would give up on trying to proof a configuration was reducible if the computer cound not do the job in 90 minutes on the IBM 360 and 30 minutes on the IBM 370. Initially there were about 2000 reducible configurations in their proof.

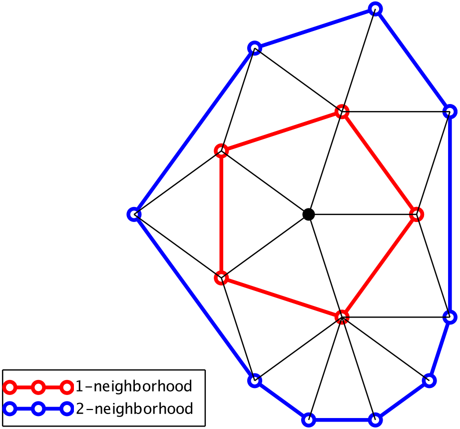
Shape conventions – Heesch



More Reducible Configurations



Think Globally – Act Locally



Four Colors Suffice

1976 (The bicentennial): Appel and Haken use 1200 hours of super computer time, and all of a Journal issue (136 pages) to prove every planar graph can be colored with 4 or fewer colors so that no two vertices of the same color are adjacent.

Paul Halmos (1990) *I do not find it easy to say what we learned from all that.*

Daniel Cohen (1991) *The mission of mathematics is understanding . . . computer shenanigans . . . leave us intellectually unfulfilled.*

Computer Errors

Compiler errors, programming errors, hardware errors:
Square root in 68020 co-processor.
Divide error in Pentium.

To only use integer arithmetic, all charges were mulitpled by 10.
This avoids all floating point issues.

Supercomputer

In the early 1970's this term was used to describe the CDC computers. In 1974 FSU had a pair CDC computers in the LOVE building basement.

The Cray and ETA super computers arrived in the late 70's. The Cray 1 (1976) ran at 80 MHz. Cray 2 (1985). FSU got an ETA in late 1980's

Origin of Supercomputing – 1929 world's fair – custom-built tabulators that IBM build for Columbia Univ

Ultrich Schmidt

Found an error in the discharge theory 1980's
Appel and Haken wrote a response.
Robertson, Sanders, Seymour and Thomas decided to derive their own proof rather than understand Appel and Haken.

Other theorems proved by Computer

1. Kepler conjecture (1611) – Optimal sphere packing in a box – 1998 referees 99% certain – part of Hilberts eighteenth problem
2. Wikipedia lists 13 results including Rubik's Cube, Sudoku, Connect Four
3. Erdös discrepancy problem (special case) – 2014 – Later Terence Tao without computers
4. Boolean Pythagorean triples problem – 2016 – yes $(1, \ldots 7824)$ but no $(1 \ldots 7825)$ – 200 terabyte proof 4 years of CPU time in 2 days on 10^5 cores.
5. Robbins conjecture – 1996 McCune & EQP – Later Mann rewrote the proof in 14 page paper.

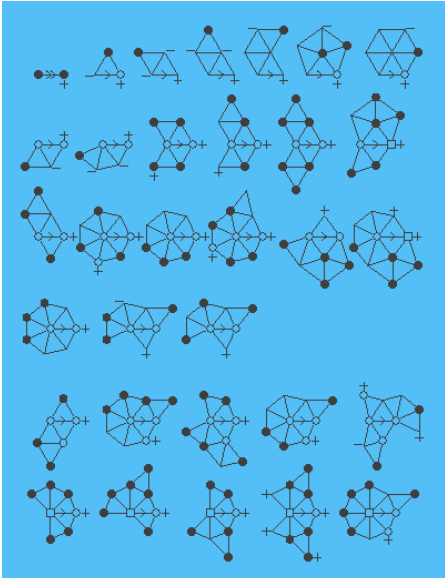
Four color solution 20 years later

Moore's law; computer speed doubles every 18 months. So 21 years is $2^{14} \sim 1600$ faster,
One technical issue was finessed.

1995: 20 years later

	Appel-Haken-Koch	Robertson-Sanders-Seymour-Thomas
Number of secondary discharging rules	486	20
Number of unavoidable configurations	1478	633
Computer time to prove	1200 hours	24 hours
Computer time to verify	Not available	5 minutes
Speed of graph coloring algorithm	$O(n^4)$	$O(n^2)$
Number of pages in final publication	741	43

Robertson, et al Rules



iMac 2.7 GHz from 2011: 3m 40s

```
This has ring-size 12, so there are 88574 colouring
and 124542 balanced signed matchings.

There are 3666 colourings that extend to the config

sfb 802> date > timer; ./reduce > /dev/null ; date
sfb 803> cat timer
Mon Oct 23 06:41:37 PM EDT 2017
Mon Oct 23 06:45:17 PM EDT 2017
```

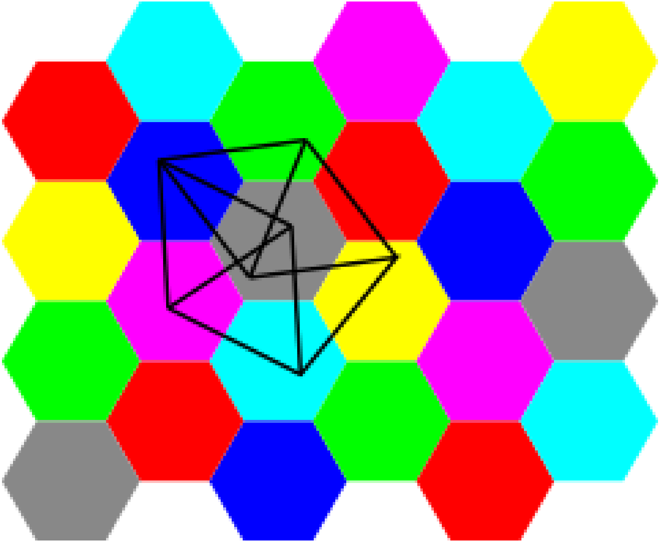
Coloring the Plane

The vertices of the plane graph are the points of the plane. Two vertices v and w are adjacent if $\|v - w\| = 1$.
What is χ the chromatic number of this graph? Known
$4 \leq \chi \leq 7$

$$\chi = \sup\{\chi(G) : G \text{ is a finite subgraph}\}\text{Axiom of Choice}$$

Projected solution date is 2084, assuming it takes as long as 4CT.

$4 \leq \chi \leq 7$



five or six?

If the color sets are defined by Jordan Curves, then $\chi \geq 6$ (Woodall 1973).
If the color sets are measureable, then $\chi \geq 5$ (Falconer 1981).
There is a model of set theory, (Solovay) where the Axiom of Choice is false but all sets are measurable.