Something for nothing

Or how to reconstruct a function from its zeros

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The Principle (naive version)

If f and g have the same zero's, then f/g is constant.

- **Multiplicities: require $\lim_{x\to c}\frac{f(x)}{g(x)} \neq 0$ at each zero c so that $\frac{f}{g}$ is continuous and never zero.**
- **Singlarity problems: $f(x)/(1+x^2)$.**
- **Growth problems: $e^x f(x)$.**
- **Non-naive form for entire functions, Hadamard 1893, but dates to at least Euler 1735.**
- **cases: polys, sin, gamma, and zeta.**

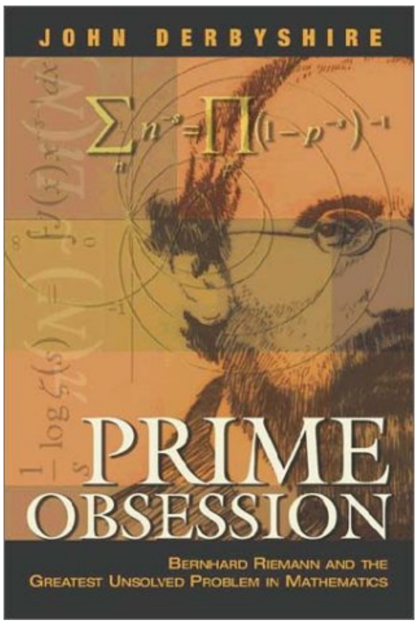
Guess the polynomial

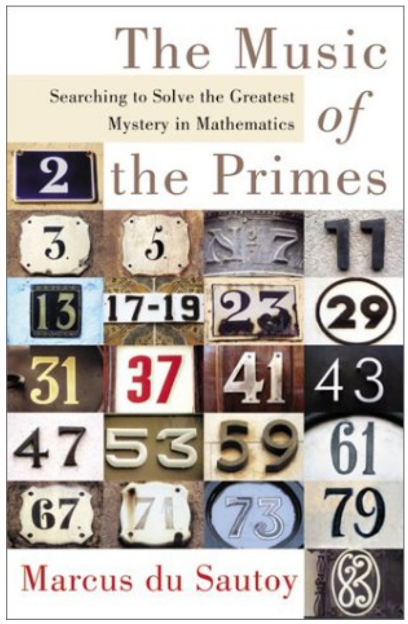
- $p(x)$ **has degree 2** $\qquad ax^2 + bx + c$
- **and** $p(2) = 0$ $\qquad (x-2)(px+q)$
- **and** $p(-1) = 0$ $\qquad k(x+1)(x-2)$
- **and** $p(0) = 6$ $\qquad -3(x-2)(x+1)$
- $p(x) = 6(1 - \frac{x}{-1})(1 - \frac{x}{2})$

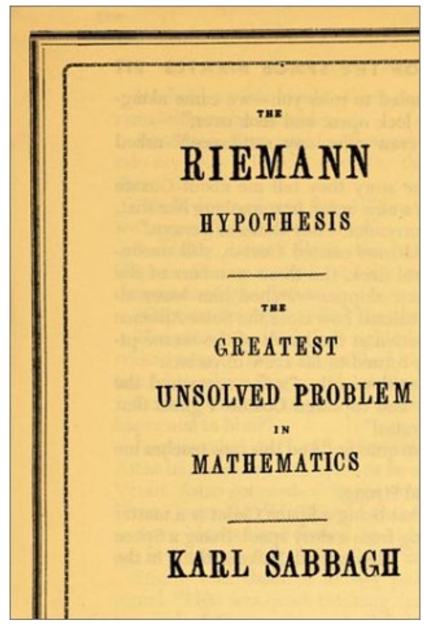
Knowing nothing is almost everything

- $c_1, c_2, \ldots c_n$ **are the zeros of a polynomial** $p(x)$**, and** $x = 0$ **is not a zero, then**
- $p(x) = p(0)(1 - \frac{x}{c_1})(1 - \frac{x}{c_2}) \cdots (1 - \frac{x}{c_n})$.
- **If polynomial** $q(0) = 0$**, and** 0 **is a root** k **times, then** $p(x) = q(x)/x^k$ **is non-zero at** 0 **and**
- $q(x) = x^k p(0)(1 - \frac{x}{c_1})(1 - \frac{x}{c_2}) \cdots (1 - \frac{x}{c_n})$
- $p(x) = p(0) \prod (1 - \frac{x}{c_i})$.

The Greatest Unsolved Problem: the Riemann Hypothesis

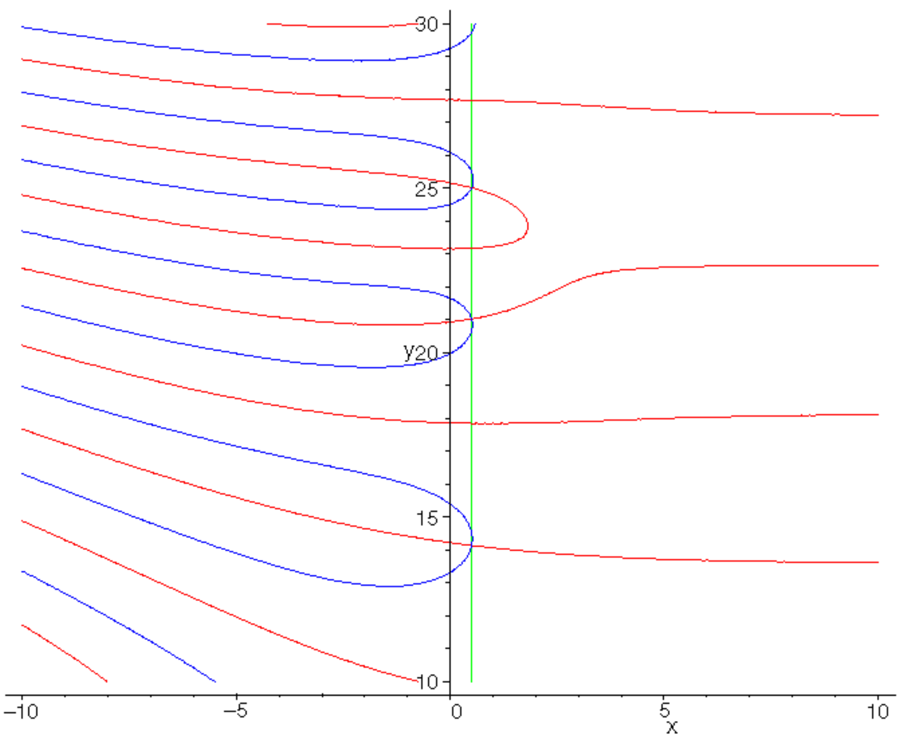






The popular press has declared the Riemann Hypothesis the greatest unsolved problem in mathematics.

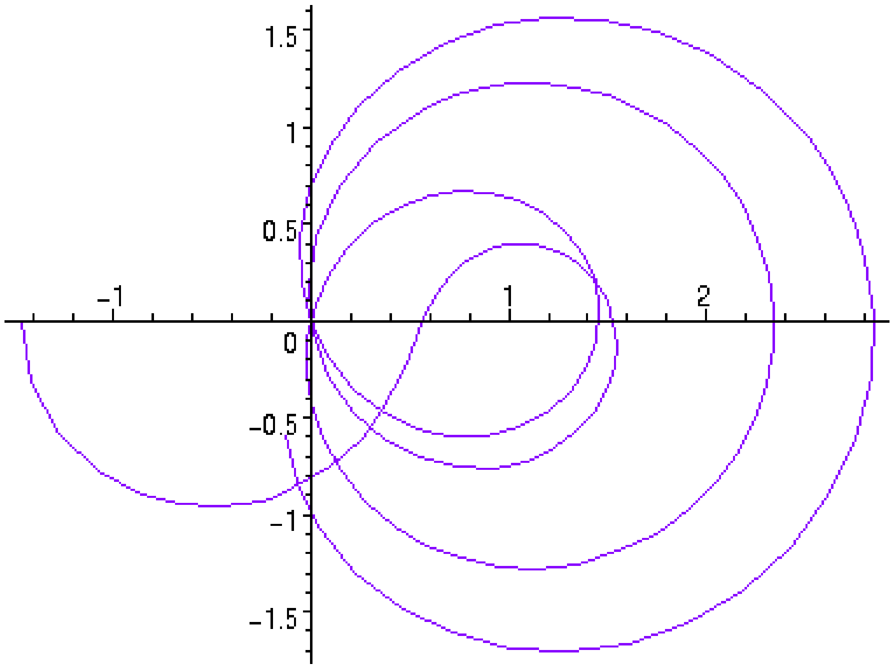
RH: ζ zeros all on the critical line



$$\psi(x) = x + \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n} \frac{x^{-2n}}{2n} + \frac{\zeta'(0)}{\zeta(0)}$$

$$\psi(x) = \sum_{p^n \le x} \ln p; \qquad |x^{s+\sigma i}| = |x^s||e^{i\sigma \ln x}| = |x^s|$$

Walking the critical line



As p runs over primes and ρ runs over non-trivial zero's of ζ.

$$\zeta(s) = g(s) \prod_{\rho} (1 - \frac{s}{\rho})$$

$$\zeta(s) = \sum_{n} \frac{1}{n^s} = \prod_{p} (1 - \frac{1}{p^s})^{-1}$$

The Basel Problem

$$\sum_n \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = ?$$

It is almost

$$\sum_n \frac{1}{n(n+1)} = \sum_n (\frac{1}{n} - \frac{1}{n+1})$$

$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots = 1$$

Solved by Euler in 1735. Famous problem popularized by Jakob Bernoulli 1689 but dates from before.

$\zeta(2)$ converges

$$\frac{1}{n(n+1)} \leq \frac{1}{n^2} \leq \frac{1}{(n-1)n}$$

$$\sum \frac{1}{n(n+1)} \leq \sum \frac{1}{n^2} \leq 1 + \sum \frac{1}{n(n+1)}$$

$$1 \leq \sum \frac{1}{n^2} \leq 2$$

Euler had a better estimates — in fact he showed

$$\sum \frac{1}{n^2} = \sum \frac{1}{2^n n^2} + (\log 2)^2$$

Lets make $\sin$ a honorary polynomial

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \text{zero is a single root}$$

$$\frac{\sin(\pi x)}{\pi x} = 0 \quad \text{for} \quad x = \pm 1, \pm 2, \pm 3 \ldots$$

$$\sin(\pi x) = \pi x \prod (1 - \frac{x^2}{n^2})$$

$$1 - \frac{x^2}{n^2} = (1 - \frac{x}{n})(1 - \frac{x}{-n})$$

Series for the product

$$
\begin{aligned}
\prod &= (1-\frac{x^2}{1^2})(1-\frac{x^2}{2^2})(1-\frac{x^2}{3^2})(1-\frac{x^2}{4^2})\cdots \\
&= 1 - x^2(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots) \\
&\quad + x^4\Big(\frac{1}{1^2}(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots) \\
&\qquad + \frac{1}{2^2}(\frac{1}{3^2}+\frac{1}{4^2}+\cdots) \\
&\qquad + \frac{1}{3^2}(\frac{1}{4^2}+\cdots)+\cdots\Big)+\cdots
\end{aligned}
$$

Basel solved, coefficents of x^3

$$\sin \pi x = \pi x \prod (1 - \frac{x^2}{n^2})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\sin \pi x = \pi x - \frac{\pi^3 x^3}{3!} + \frac{\pi^5 x^5}{5!} - \cdots$$

$$-\frac{\pi^3}{3!} = -\pi \sum_n \frac{1}{n^2}$$

so $$\sum_n \frac{1}{n^2} = \frac{\pi^2}{6}$$

Bernoulli's Objections

- **Infinite products?**
- **What about complex zeros of $\sin \pi x$?**
- **What about $e^x \sin(\pi x)$? (Has the same zeros)**
- **Euler 1735 thought this was a wonderful technic and as good as any other solution. So did Riemann 1859. Hadamard actually proved the technic in 1893.**

Infinite products

- $\prod_{n=1}^{N}(1-a_n)$ **converges** $\iff$ **its** $\log$ **converges.**

- **The tangent approx** $\log(1-x) \approx -x$**.**

- $\sum_{n=1}^{N}\log(1-a_n)$ **converges** $\iff$ $\sum_{n=1}^{N}a_n$ **converges**

- **So the RHS,** $\pi x \prod(1-\frac{x^2}{n^2})$ **converges since** $\sum \frac{x^2}{n^2}$ **converges. But does it converge to** $\sin(\pi x)$**?**

- $\log\zeta(s)$ **can be similarly written as function of its zeros.**

Zero hocus pocus I

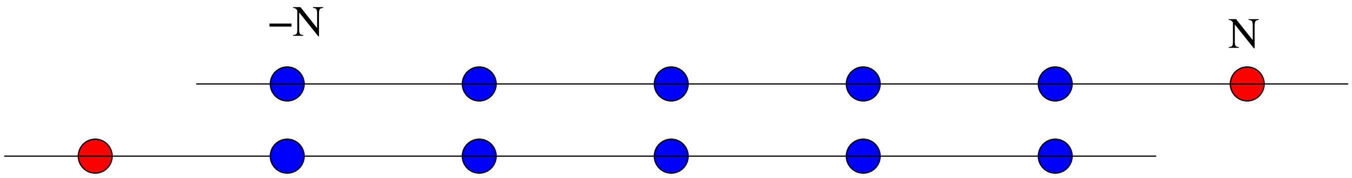
$\sin x$ **has zeros at** $n\pi$**,** $\sin(x - \pi/2)$ **has zeros at** $n\pi/2$ **for odd** n**, so** $\sin x \cos x$ **has zeros at** $n\pi/2$**.**

But $\sin 2x$ **has zeros at** $n\pi/2$ **so if you could use the zero's to determine a function, then** $\sin 2x = k \sin x \cos x$ **some constant** k**.**

But this is TRUE. It is the double angle formula $\sin 2x = 2 \sin x \cos x$**.**

Translation

$$g_N(x) = \pi x \prod_{n=-N}^{N} (1 - \frac{x}{n})$$

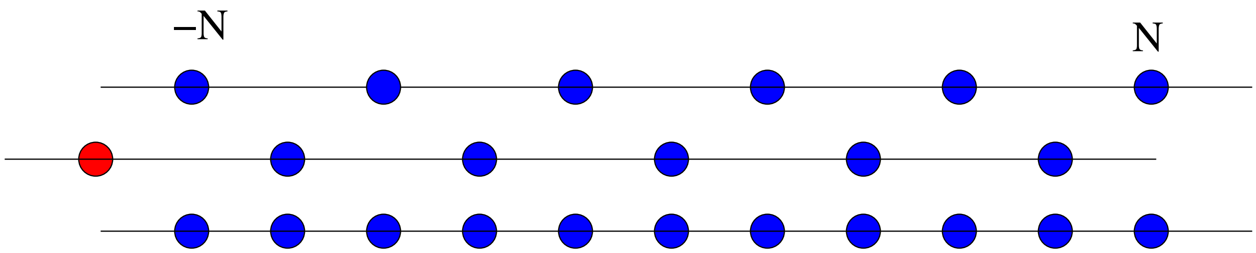


Compare factors of $g_N(x)$ and $g_N(x+1)$. The two red balls don't match but the others line up with $(1 - \frac{x}{n-1})$ and $(1 - \frac{x+1}{n})$ having a zero at $n-1$ and ratio $\frac{n}{n-1}$ which exactly cancels the ratio at $-n$. Fix x and let $N \gg |x|$, then the two red factors are ≈ 1 at x. Thus $g(x) = \lim g_N(x)$ has period 1

$$g(x) = g(x+1)$$

Dilation

$$g_{2N}(2x) = 2\pi x \prod_{n=-2N}^{2N} (1 - \frac{2x}{n})$$



Separate the product into even and odd n **we almost get** $2g_N(x)g_N(x+\frac{1}{2})$ **missing only by the red ball. When** $N \gg |x|$**, red factor** ≈ 1 **as before, thus** $g(x) = \lim g_N(x)$ **satisfies the fun eqn**

$$g(2x) = 2g(x)g(x + \frac{1}{2})$$

Zero hocus pocus II

Let $g(x) = xe^{\gamma x} \prod e^{-x/n}(1 + x/n)$**. Note** $g(x)$ **has zeros at** $0, -1, -2, \cdots$ **and**

$$g(1) = 1$$

Note $xg(x + 1)$ **also has the same set of zeros and the same value at** $x = 1$**, then**

$$xg(x + 1) = g(x)$$

Since $g(x)$ **is zero at** $0, -1, -2, \cdots$**, then** $g(1 - x)$ **is zero at** $1, 2, 3, \cdots$**, then**

$$g(x)g(1 - x) = k \sin(\pi x)$$

$\Gamma(x) = 1/g(x)$**, these formula translate to** $\Gamma(x + 1) = x\Gamma(x)$ **so that** $\Gamma(n + 1) = n!$ **and** $\Gamma(x)\Gamma(1 - x) = \pi/\sin(\pi x)$

Cot and the Herglotz Trick

$$\pi \cot \pi x = \frac{1}{x} + \sum_n \left(\frac{1}{x+n} + \frac{1}{x-n}\right) \; x \notin Z$$

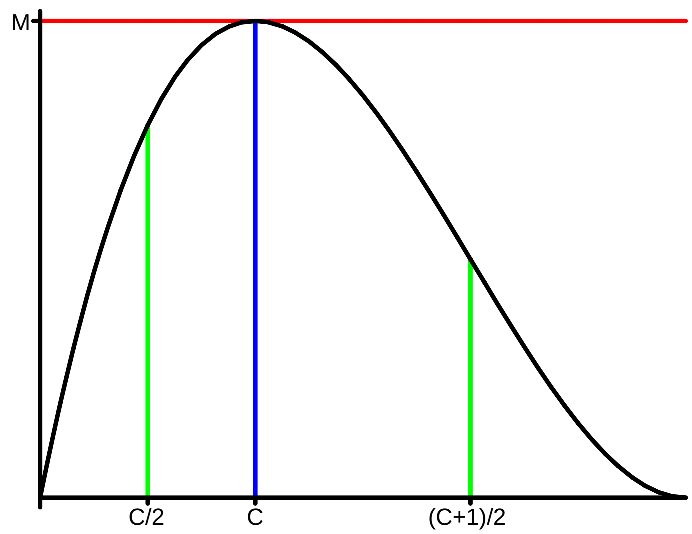
Let $f(x)$ be rhs, and let

$$g(x) = \lim_N \sum_{n=-N}^{N} \frac{1}{x+n}$$

Claim both f and g are (i) continuous off Z; (ii) have period 1; (iii) odd and (iv) satisfy the functional equation

$$F\left(\frac{x}{2}\right) + F\left(\frac{x+1}{2}\right) = 2F(x)$$

Let $h(x) = f(x) - g(x)$. Extend $h(n) = 0$, then h is continuous, odd, periodic and satisfies the functional equation. It follows that h is identically zero.



Let M be the maximum value of h and suppose c is so that $h(c) = M$. Both $h(\frac{c}{2})$, $h(\frac{c+1}{2}) \leq M$ and the fun eqn says their average is M, so $h(\frac{c}{2}) = M$. Iterating $0 = \lim_n h(\frac{c}{2^n}) = M$.