Assignment 1 (Graph Theory and Networks)
Discussed in class on January 24

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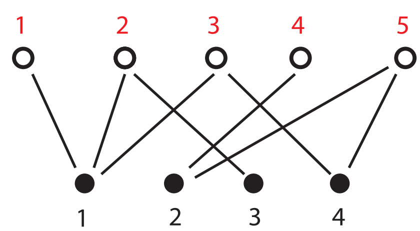
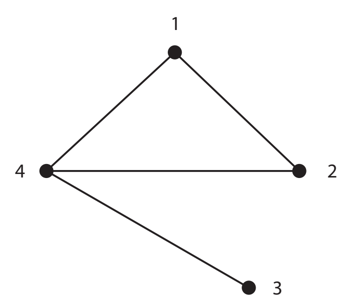
(1) Consider an undirected network of n nodes. What is the maximum number of edges this network can have? Given a single edge, what is the maximum number of node pairs that the edge can connect? Given a single node, what is the maximum number of edges that can connect to that node?

(2) Consider the bipartite network below, where the top row of nodes (open) are groups and the bottom row (filled) are actors.
   (a) Give the incidence matrix $\mathbf{B}$.
   (b) What does $\mathbf{B}_{31} = 1$ mean? How about $\mathbf{B}_{13} = 0$?
   (c) Why is $\mathbf{B}$ not square?
   (d) Give the two one-mode projections of the bipartite network, and their adjacency matrices $\mathbf{A}$ (for actors) and $\mathbf{G}$ (for groups).
   (e) What does $\mathbf{A}_{31} = 1$ mean? How about $\mathbf{G}_{13} = 1$?
   (f) Why are $\mathbf{A}$ and $\mathbf{G}$ square and symmetric?



(3) Can you recover a bipartite graph using its two one-mode projections? If so, then the one-mode projections should map back to a single bipartite graph. Check this by constructing a bipartite graph with four groups and five actors, and such that each group and actor has at least one edge. Place the edges so that two different bipartite graphs are formed, but with the same one-mode projections. If you can do this, then it means that information is lost when moving from the two-mode graph to its one-mode projections, and you therefore cannot recover a unique bipartite graph from its two one-mode projections.

(4) Answer the following questions for the graph below:

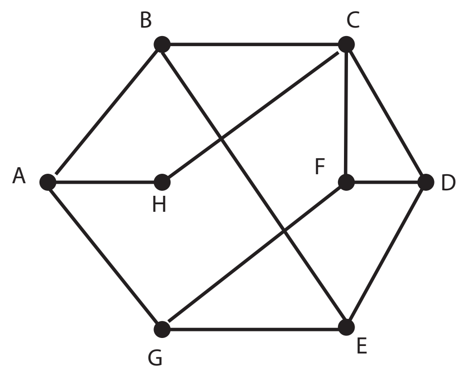


(a) How many paths of length 1 are there from node 2 to 4? List them.
(b) How many paths of length 2 are there from node 2 to 4? List them.
(c) How many paths of length 3 are there from node 2 to 4? List them.
(d) Show that the path cardinality you gave above matches the appropriate element of the matrix $\mathbf{A}^k$ for appropriate k.
(e) Compute the number of triangles using the trace of the appropriate matrix. Does this agree with what you see?

(5) Consider an undirected network with n vertices and adjacency matrix $\mathbf{A}$. Suppose a path of length r has weight α^r.
(a) What is the total number of paths of length r between two vertices s and t?
(b) What is the sum of the weights of paths of length r between s and t?
(c) What is the sum of the weights of all paths of length r or less between s and t?

(d) What does this last sum converge to in the limit $r \to \infty$? (Hint: you did this in calculus, but with a scaler variable x rather than a matrix)

(6) On the network below, use a breadth-first search to determine the distance between node A and every other node. Then list the geodesic (i.e., shortest) paths between node A and each node in the network. Repeat for node F.



(7) Consider a tree with 10 nodes and in which five nodes have degree 1, three have degree 2, one has degree 3, and one has degree 4.
(a) Illustrate the tree in a rooted fashion.
(b) How many edges are there?
(c) If you add an edge is the network a tree?
(d) How about if you take an edge away, is the network still a tree?