Assignment 2 (Graph Theory and Networks)

Discussed in class on February 14

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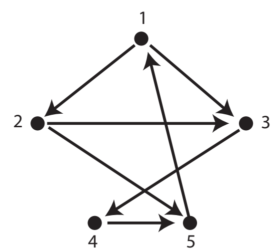
(1) The node degrees can be collected into a vector $\vec{k}$.
  (a) Suppose that the degree vector for a simple undirected (and unweighted) graph is $\vec{k} = (2, 2, 3, 3, 2)$. Give a plausible adjacency matrix **A**.
  (b) Next consider a simple directed graph with in-degree and out-degree vectors $\vec{k}^{\text{in}} = (3, 3, 2, 3, 2)$ and $\vec{k}^{\text{out}} = (3, 3, 3, 3, 1)$. Give a plausible adjacency matrix **B**.
  (c) What is the density of each of these graphs?

(2) Give one real-life example (not given in class) of each of the following types of networks:
  (a) An acyclic directed graph
  (b) A cyclic directed graph
  (c) A tree (but not an actual tree or plant)
  (d) A planar network
  (e) A bipartite network

(3) Consider a bipartite network, with two types of nodes, and suppose that there are n_1 nodes of type 1 and n_2 nodes of type 2. Show that the mean degrees c_1 and c_2 of the two types are related by

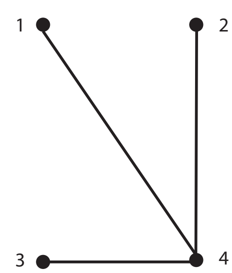
$$c_2 = \frac{n_1}{n_2} c_1$$

(4) Consider the network below. Why is it not a proper citation network? Remove one edge to make it a proper citation network, then do the following.
  (a) Give the adjacency matrix (**A**).
  (b) Give the bibliographic coupling matrix (**B**).
  (c) Give the cocitation matrix (**C**).
  (d) What does $\mathbf{B}_{12} = 1$ mean? How about $\mathbf{B}_{45} = 0$, $\mathbf{B}_{11} = 2$, $\mathbf{B}_{33} = 1$?
  (e) What does $\mathbf{C}_{25} = 0$ mean? How about $\mathbf{C}_{35} = 1$?



(5) Can you determine the bibliographic coupling network solely from the cocitation network? In terms of matrices, can you determine the adjacency matrix for bibliographic coupling, **B**, from the cocitation adjacency matrix, **C**? If yes, explain how. If no, give an example.

(6) Consider diffusion on the graph shown below, with diffusion coefficient c. Suppose that the diffusing substance has initial distribution $\vec{u}(0) = (4, 3, 2, 7)$.
   (a) Write down the graph Laplacian.
   (b) What is the value of the smallest eigenvalue, λ_1, and its corresponding eivenvector $\vec{v}_1$?
   (c) What is the initial value of the first coefficient of the spectral solution, $a_1(0)$?
   (d) What is the equilibrium distribution of substance across the graph?



(7) The World Wide Web has a bowtie structure, which is illustrated in this problem. Consider the set of eighteen Web pages shown in the figure below, with links forming a directed graph.

(a) Which set of nodes constitutes the largest strongly connected component (SCC) in this network?

(b) Taking this as the *giant SCC*, which nodes represent Web pages that have some path into the giant SCC, but are not themselves in the giant SCC? (These constitute the IN set.)

(c) Which nodes represent Web pages that the giant SCC links to from some path, but are not part of the giant SCC? (These constitute the OUT set.)

(d) *Tendrils* consist of nodes not in the giant SCC, IN, or OUT, but which (i) are reachable from IN and cannot reach the giant SCC and (ii) can reach OUT but cannot be reached from the giant SCC. In the network below, which nodes constitute the tendrils?

(e) Name an edge you could add or delete so as to increase the size of the giant SCC.

(f) Name an edge you could add or delete so as to increase the size of IN.

(g) Name an edge you could add or delete so as to increase the size of OUT.