1. Some asymptotic expansion problems
For small $\epsilon$, determine two terms in the asymptotic expansion of each root of the following equations:

$\begin{align*}
(1) \quad x^2 - 2x + (1 - \epsilon^2)^{25} &= 0 \\
(2) \quad \epsilon x^3 + x - 2 &= 0
\end{align*}$

2. Dynamics on a line
For the following, sketch the phase portrait (on the line), find all steady states and determine their stability from the velocity function. Then sketch a graph of the $x(t)$ time course for several initial conditions.

$\begin{align*}
(1) \quad \dot{x} &= x^4 - x^2 \\
(2) \quad \dot{x} &= e^x - \cos x. \quad \text{(Hint: Sketch the graphs of } e^x \text{ and } \cos x \text{ on the same axes, and look for intersections. You won’t be able to find the fixed points explicitly, but you can still determine the qualitative behavior.)}
\end{align*}$

3. Finding and working with potential functions
Find and sketch a potential function $V(x)$ for $\dot{x} = \sin x + x \cos x$. Use this to construct a phase portrait, including several positive and negative equilibria. In terms of the potential function, explain why oscillations are impossible (this same argument applies to any ODE with a potential function).

4. Bifurcation analysis problems
Analyze the following for the full range of parameter values (assume that the parameter $r$ can take on all real values). In each case, sketch a phase portrait, determine the locations and stability of steady states, and sketch a bifurcation diagram. Indicate the locations and types of all bifurcations.

$\begin{align*}
(1) \quad \dot{x} &= rx - \frac{x}{1+x^2} \\
(2) \quad \dot{x} &= rx - \ln(1 + x)
\end{align*}$

5. An ODE with an infinite number of bifurcations
Consider the ODE $\dot{x} = rx + \sin x$, with parameter $r > 0$. Determine approximate values of $r$ where bifurcations occur. What type of bifurcations are they?