1. More Lindstedt-Poincaré
This problem uses the Lindstedt-Poincaré method on a nonlinear ordinary differential equation. At some point you’ll want to use the following trig identity: \( \sin^2 \theta = \frac{1}{2} (1 - 2 \cos \theta) \).

Consider the equation

\[ \ddot{x} + \omega_o^2 x = \epsilon \dot{x}^2 x. \]

where \( \omega_o \) is the frequency of the reduced-system oscillator.

1. Determine a first-order uniform asymptotic approximation to the solution using the Lindstedt-Poincaré technique.
2. Find a solution that satisfies the initial conditions \( x(0) = 0, \dot{x}(0) = \beta \).

2. A conservative system
Find a conserved quantity for the system \( \ddot{x} = a - e^x \), and sketch phase portraits for \( a < 0, a = 0, \) and \( a > 0 \).

3. A curve of Hopf bifurcations
Consider the biased van der Pol oscillator \( \ddot{x} + \mu (x^2 - 1) \dot{x} + x = a \). Find the curves in \( (\mu, a) \) space at which Hopf bifurcations occur.

4. A neat bifurcation diagram
Construct a bifurcation diagram for the system

\[
\begin{align*}
\dot{r} &= r(\mu - \sin r) \\
\dot{\theta} &= 1
\end{align*}
\]

as \( \mu \) varies. Classify all bifurcations.

5. Finding a trapping region
In this problem we will prove the existence of a limit cycle by finding a trapping region. Consider

\[
\begin{align*}
\dot{x} &= x - y - x(x^2 + 5y^2) \\
\dot{y} &= x + y - y(x^2 + y^2)
\end{align*}
\]

1. There is a single equilibrium, at the origin. Classify this equilibrium.
(2) Show that this system is equivalent to the following system in polar coordinates:

\[
\begin{align*}
\dot{r} &= r[1 - r^2 - r^2 \sin^2(2\theta)] \\
\dot{\theta} &= 1 + 2r^2 \sin(2\theta) \sin^2 \theta
\end{align*}
\]

by using \( r\dot{r} = x\dot{x} + y\dot{y} \) and \( \dot{\theta} = (x\dot{y} - y\dot{x})/r^2 \) and some trig identities.

(3) Determine a trapping region to establish the existence of a periodic orbit.