Bifurcations

What is it?

Def: A bifurcation is a qualitative change that occurs in the dynamics of a dynamical system in response to a change in the value of a parameter.

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x};\mu)$$

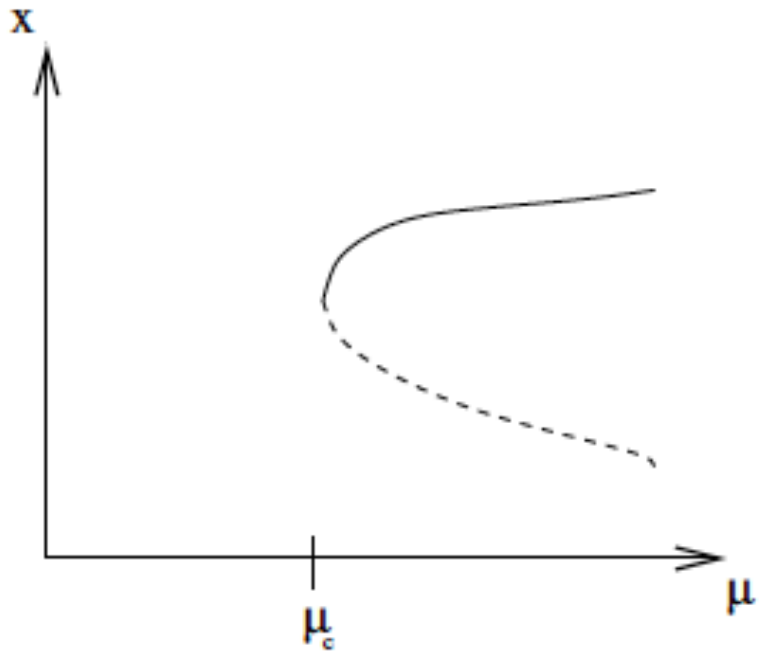
The vector contains the state variables, while μ is a parameter.

$$\frac{d\vec{x}}{dt} = \mathbf{J}\vec{x}$$

Much can be learned about the asymptotic states of the system (i.e., equilibria) by linearizing it and examining eigenvalues of the Jacobian matrix **J.**

Saddle-Node Bifurcation

Def: Emergence of two new equilibria from none.

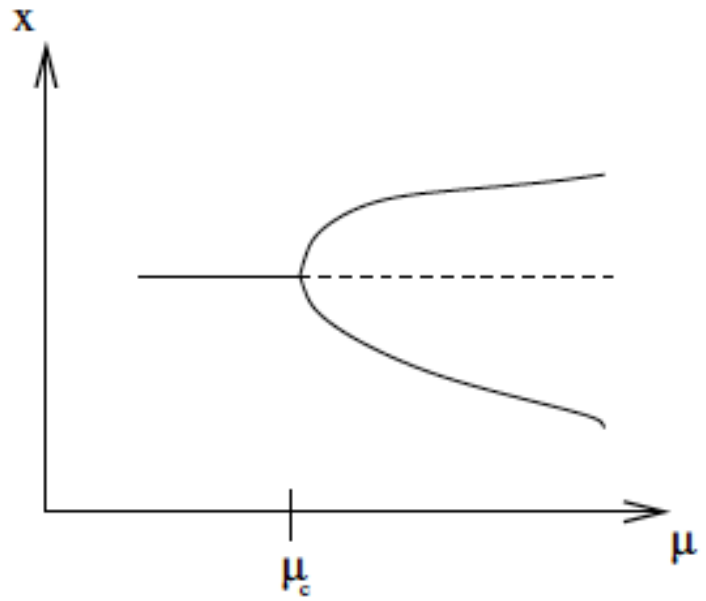


Solid: branch of stable equilibria
Dashed: branch of unstable equilibria

Saddle-node bifurcation at μ_c

Pitchfork Bifurcation

Def: A symmetry-breaking phenomenon, where a single equilibrium splits into three.

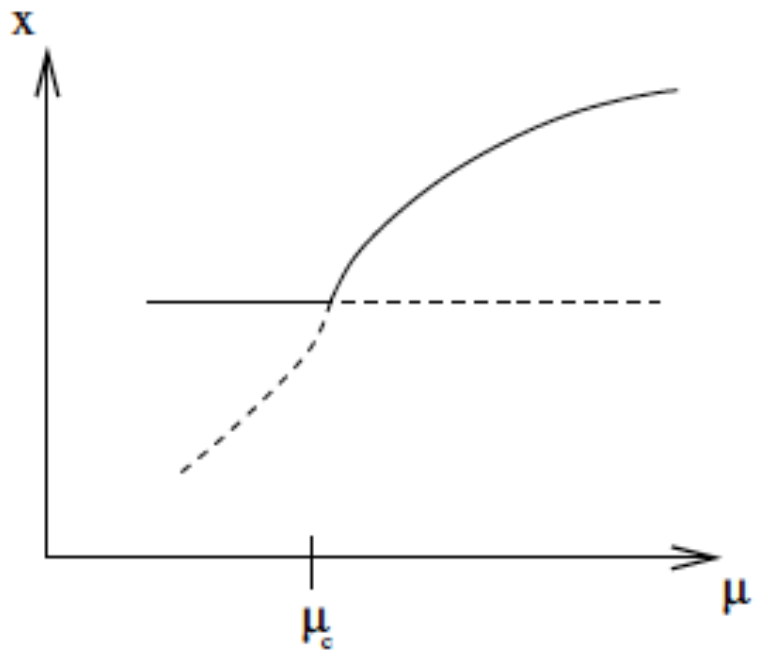


Solid: branch of stable equilibria
Dashed: branch of unstable equilibria

Supercritical pitchfork bifurcation at μ_c

Transcritical Bifurcation

Def: Stability is transferred from one branch of equilibria to another.

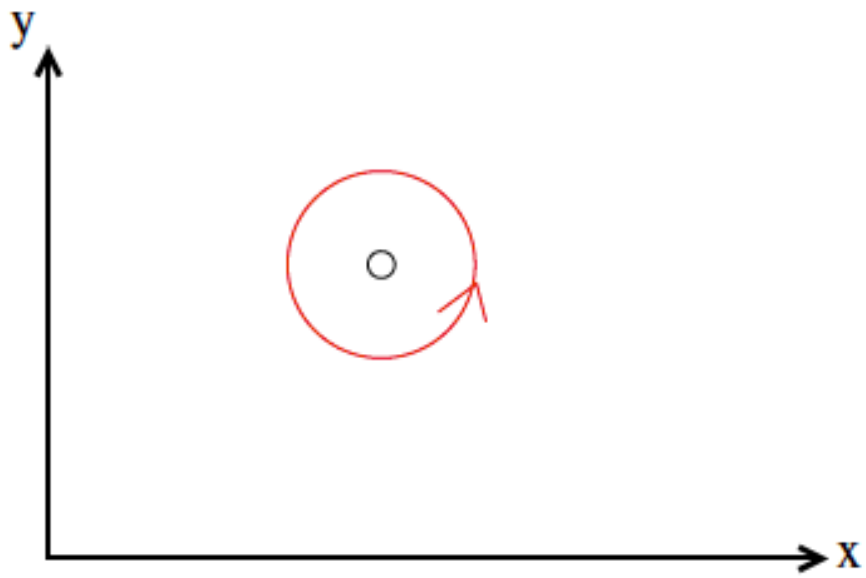


Solid: branch of stable equilibria
Dashed: branch of unstable equilibria

Transcritical bifurcation at μ_c

Hopf Bifurcation in the Phase Plane

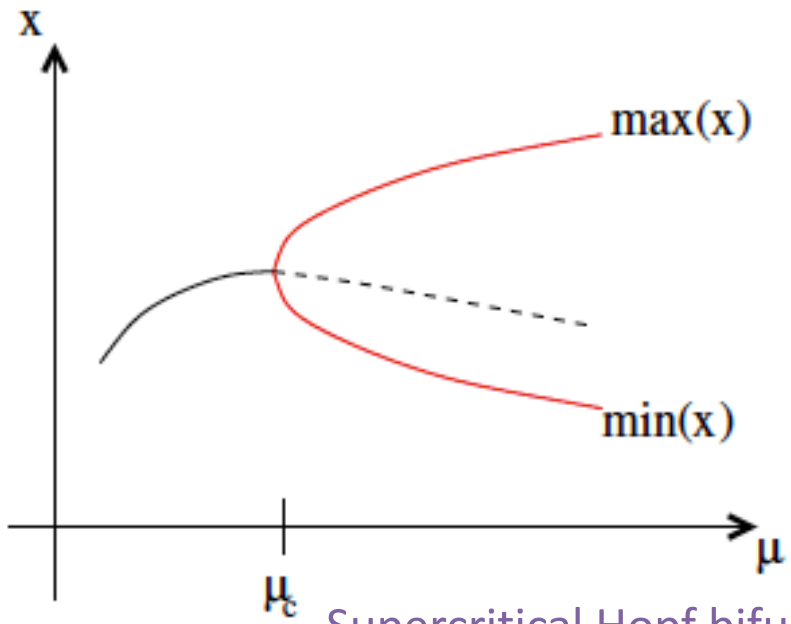
Def: A periodic solution is born out of an equilibrium as it changes stability.



Phase plane view: the periodic solution appears as a closed oriented curve, called a limit cycle.

Supercritical Hopf Bifurcation Diagram

Def: A periodic solution is born out of an equilibrium as it changes stability.



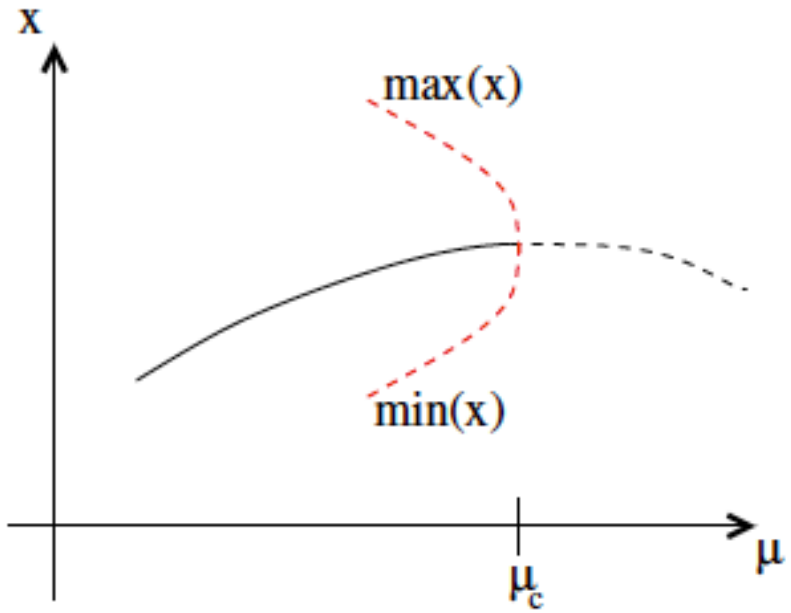
Supercritical Hopf bifurcation at μ_c

Solid black: branch of stable equilibria
Dashed black: branch of unstable equilibria
Solid red: branch of stable limit cycles

The newly-born oscillations have small amplitude

Subcritical Hopf Bifurcation Diagram

Def: A periodic solution is born out of an equilibrium as it changes stability.



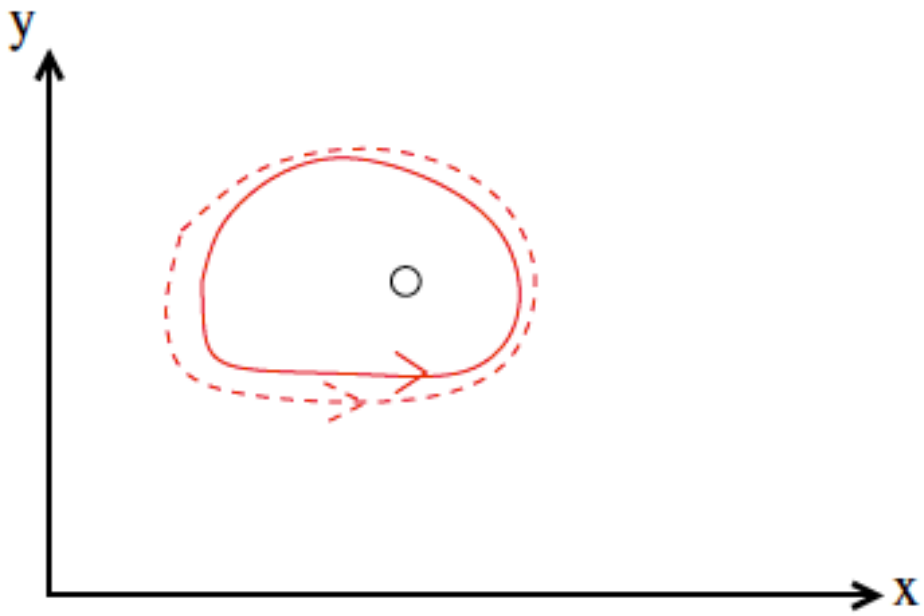
Subcritical Hopf bifurcation at μ_c

Solid black: branch of stable equilibria
Dashed black: branch of unstable equilibria
Solid red: branch of stable limit cycles

Newly-born oscillations are unstable

Saddle-Node of Periodics Bifurcation in the Phase Plane

Def: A stable and an unstable limit cycle coalesce.

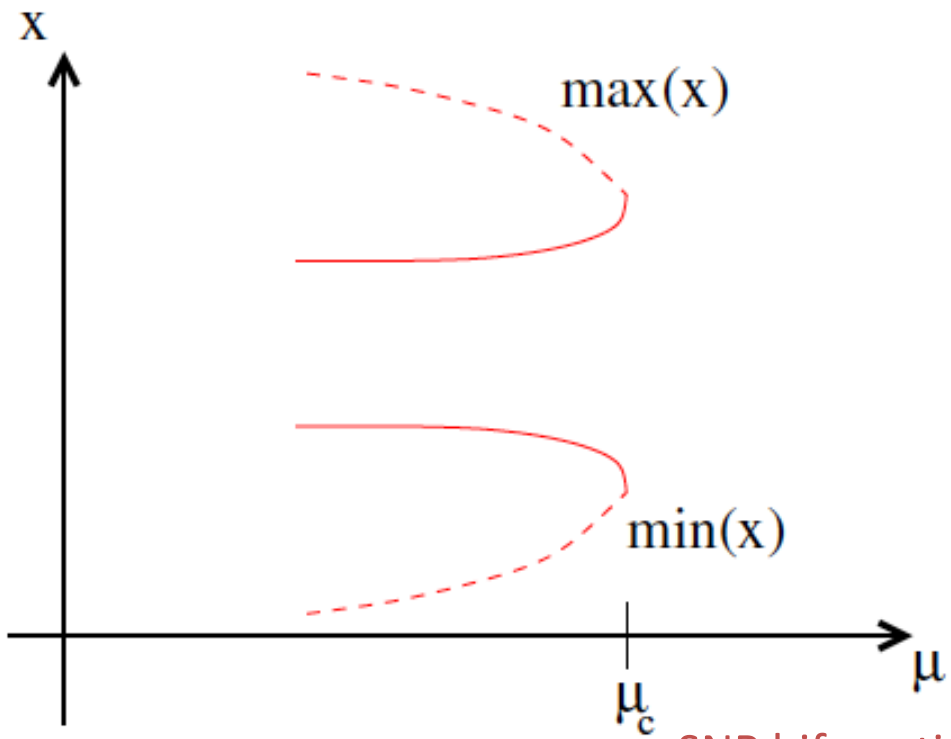


Solid red: stable limit cycle
Dashed red: unstable limit cycle

Not yet at the bifurcation, but close!

Saddle-Node of Periodics Bifurcation Diagram

Def: A stable and an unstable limit cycle coalesce.

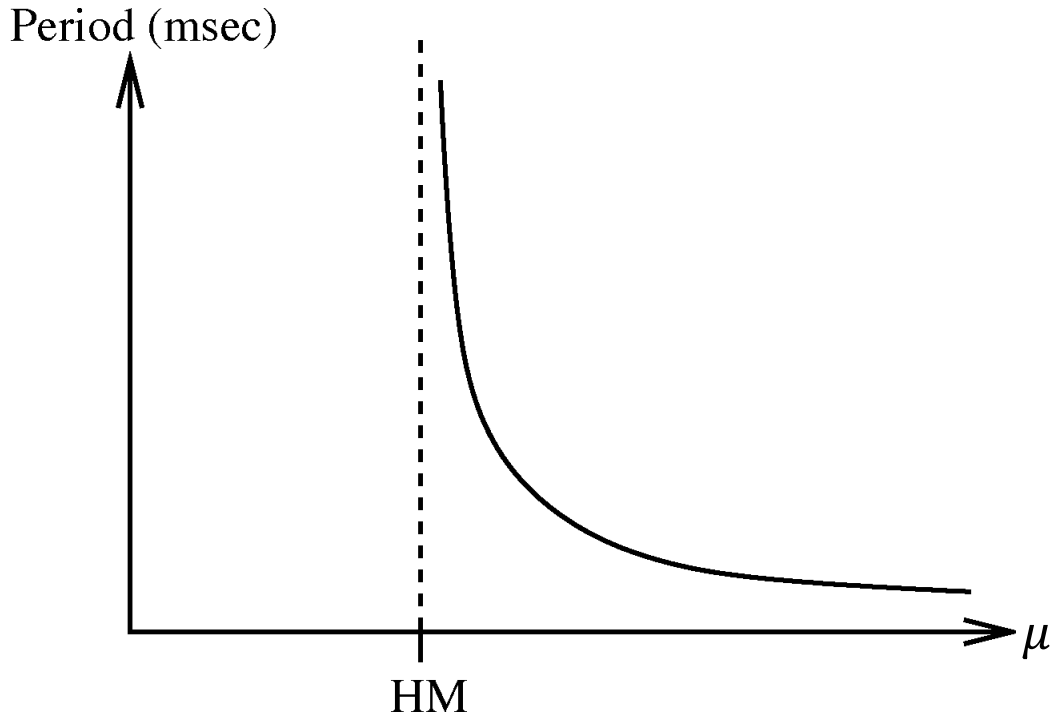


SNP bifurcation at μ_c

Solid red: branch of stable limit cycles
Dashed red: branch of unstable limit cycles

Bifurcation Diagram of Period

Def: A homoclinic orbit is an infinite-period orbit that occurs when a limit cycle connects up with a saddle point.

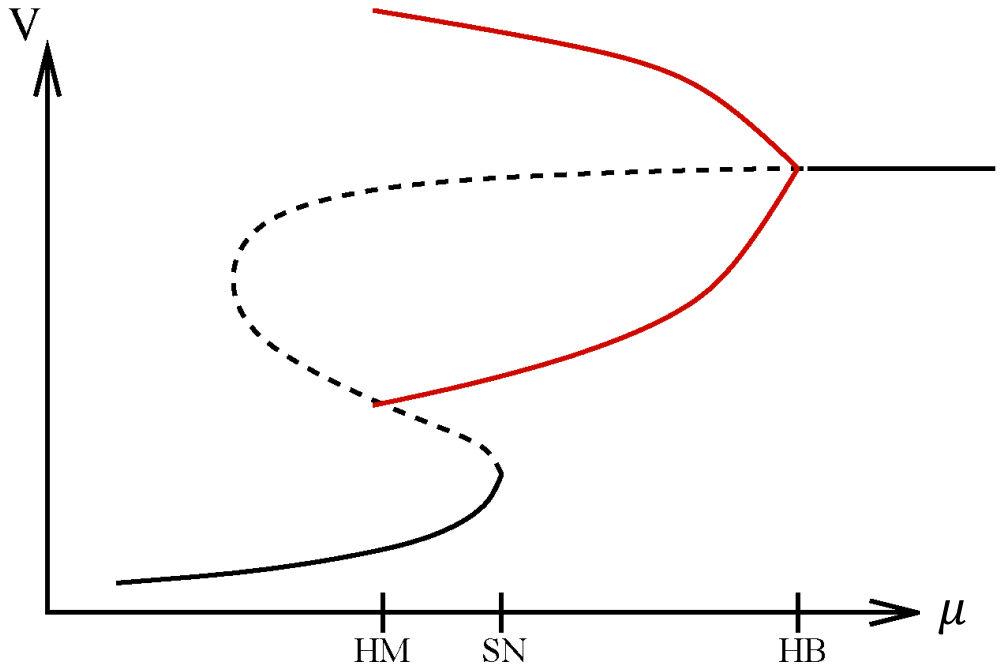


Black: period of limit cycles

Homoclinic bifurcation at μ_{HM}

Homoclinic Bifurcation Diagram

Def: A homoclinic orbit is an infinite-period orbit that occurs when a limit cycle connects up with a saddle point.



Red: branch of stable limit cycles

Black: branch of stable and unstable equilibria

Homoclinic bifurcation at μ_{HM}