Bifurcations

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A **bifurcation** is a qualitative change in the asymptotic structure of a dynamical system. Stationary bifurcations involve changes in the number and/or stability of steady states. Periodic bifurcations involve changes in the number and/or stability of periodic solutions. All of these bifurcations can occur in continuous dynamical systems of two dimensions or higher, but for simplicity we will discuss planar systems (the generalization to higher dimensions is straight forward). Consider a typical nonlinear *planar* system

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x};\mu) \tag{1}$$

where $\vec{x}$ is a 2-dimensional vector, μ is a parameter, and $\vec{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a nonlinear function. The roots of $\vec{F}$ are the steady states and their stability can be determined by *linearizing* the system about these points. Let $\vec{p}$ be a steady state of the planar system, then the linearization about $\vec{p}$ is

$$\frac{d\vec{x}}{dt} = \mathbf{J}\vec{x} \tag{2}$$

where $\mathbf{J}$ is the 2×2 **Jacobian matrix** evaluated at $\vec{p}$.

Stationary bifurcations

A steady state $\vec{p}$ is a node if the two eigenvalues λ_1 and λ_2 of $\mathbf{J}$ are real and have the same sign. If the eignenvalues are negative, then $\vec{p}$ is stable; otherwise it is unstable. The steady state is a saddle point if the eigenvalues are real with opposite signs. If the eigenvalues form a complex conjugate pair, then $\vec{p}$ is a focus (also called a *spiral*). If both eigenvalues of $\vec{p}$ have non-zero real parts, then the steady state is **hyperbolic**. This is the typical case, and by the Hartman-Grobman theorem the linearization is a good description of the flow in the neighborhood of a hyperbolic steady state. If, in contrast, $\vec{p}$ has an eigenvalue with zero real part it is called **non-hyperbolic** and the flow of the linearlized system may not be a good description of the flow of the nonlinear system near $\vec{p}$. Suppose that Eq. 1 has a non-hyperbolic equilibrium when parameter $\mu = \mu_c$. Then there is a bifurcation at this value of the parameter. *Bifurcations occur at non-hyperbolic steady states.*

At a stationary bifurcation one of the eigenvalues is zero. At a saddle-node bifurcation a saddle point and a node coalesce, creating a single steady state with a zero eigenvalue. For μ on one side of this bifurcation there are no steady states; for μ on the other side there are two steady states.

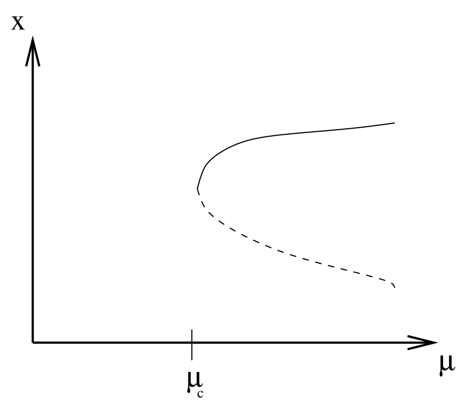

Figure 1: Saddle node bifurcation at $\mu = \mu_c$. Solid=stable, dashed=unstable.

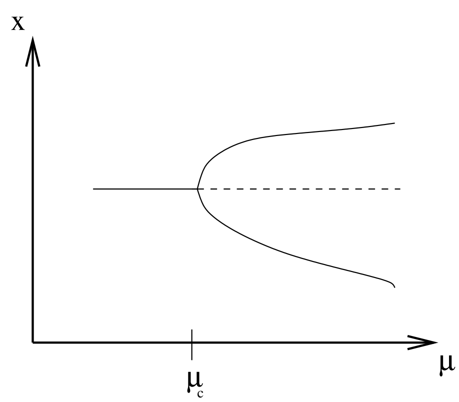

Figure 2: Pitchfork bifurcation at $\mu = \mu_c$. Solid=stable, dashed=unstable.

A pitchfork bifurcation can occur in symmetric systems. The number of equilibria goes from 1 to 3 as this bifurcation is crossed and the stability of the original equilibrium changes.

At a transcritical bifurcation the stability is transferred from one equilibrium point to another. There are two equilibria for μ on either side of μ_c, and one equilibrium when $\mu = \mu_c$. The stable equilibrium for $\mu < \mu_c$ becomes unstable for $\mu > \mu_c$ and vice versa.

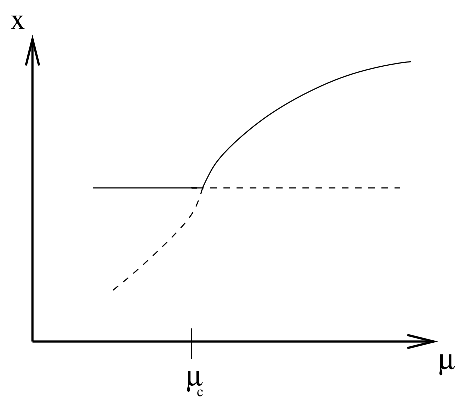

Figure 3: Transcritical bifurcation at $\mu = \mu_c$. Solid=stable, dashed=unstable.

Periodic bifurcations

A bifurcation involving the change in the number or stability of a periodic solution, or limit cycle, is a periodic bifurcation. A Hopf bifurcation is a periodic bifurcation in which a new limit cycle is born from a stationary solution. The stationary solution changes stability as the limit cycle is born. This occurs when the eigenvalues of the stationary solution cross the imaginary axis. Suppose that $\vec{p}$ is a steady state and the eigenvalues of the linearization are purely imaginary when $\mu = \mu_c$, but for some neighborhood to the left of μ_c and some neighborhood to the right of μ_c the eigenvalues are complex with non-zero real part. Then there is a Hopf bifurcation at $\mu = \mu_c$.

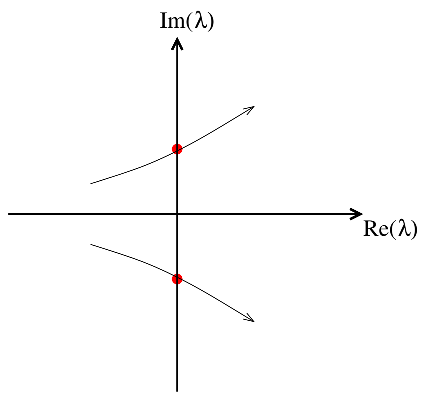

Figure 4: The eigenvalues cross through the imaginary axis at a Hopf bifurcation. The red points indicate the Hopf bifurcation.

According to the Hopf Bifurcation Theorem, the following things are true at a Hopf bifurcation:

- The limit cycle born at the bifurcation has 0 amplitude. As μ is moved away from μ_c the amplitude A grows as $A \propto \sqrt{|\mu - \mu_c|}$ for μ in some neighborhood of μ_c.

- The period T of the limit cycle born at the bifurcation is $T = \frac{2\pi}{\mathrm{Im}(\lambda)}$.

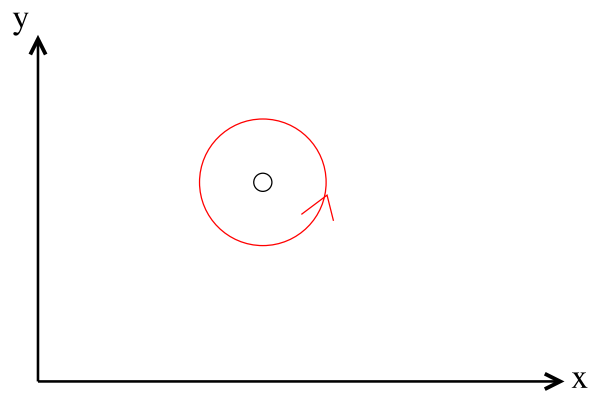

Figure 5: The new limit cycle (red) and the steady state (black) have opposite stability properties.

At a supercritical Hopf bifurcation the limit cycle that is born is stable. At a subcritical Hopf bifurcation the limit cycle is born unstable. In the associated bifurcation diagrams the periodic branch that emerges from the stationary branch at the bifurcation is stable (supercritical) or unstable (subcritical).

Another type of periodic bifurcation is a **saddle-node of periodics** (SNP) bifurcation. This occurs when a stable limit cycle coalesces with an unstable limit cycle, creating a limit cycle that attracts from one direction and repels from another (it might attract phase points inside the cycle and repel those

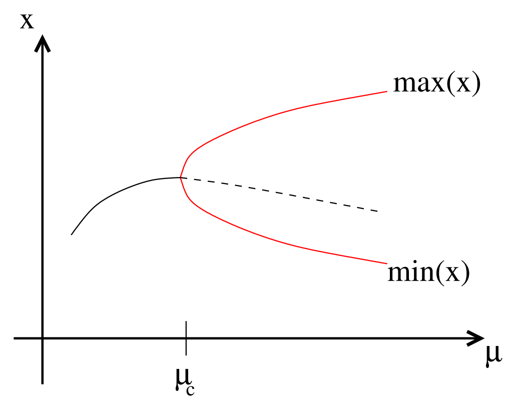

Figure 6: Typical bifurcation diagram near a supercritical Hopf bifurcation. Black=stationary branch, Red=min and max of stable periodic branch.

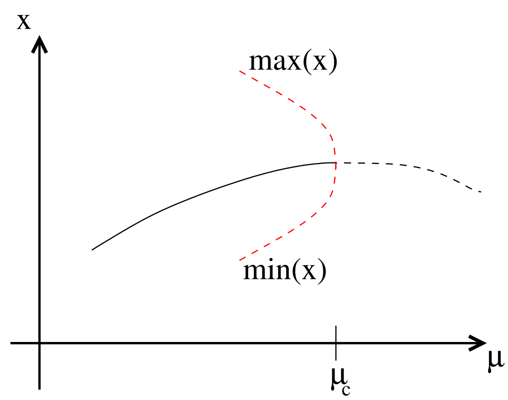

Figure 7: Typical bifurcation diagram near a subcritical Hopf bifurcation. Black=stationary branch, Red=min and max of unstable periodic branch.

outside, or vice versa). As the parameter μ is moved past the bifurcation point μ_c the limit cycle disappears.

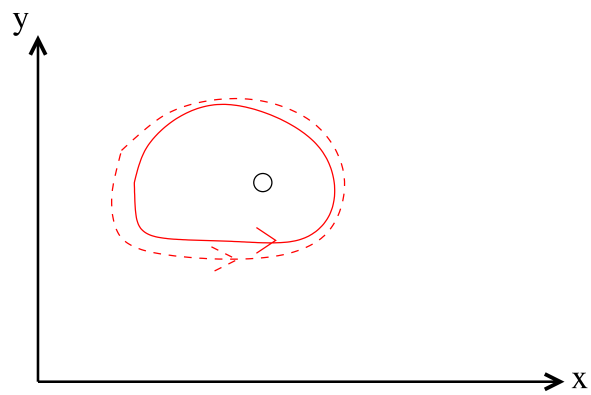

Figure 8: Two limit cycles near a saddle-node of periodics (SNP) bifurcation.

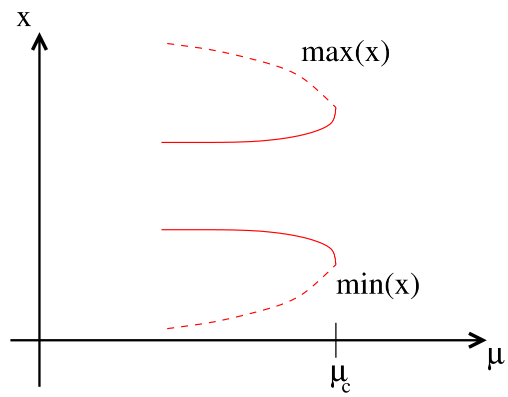

Figure 9: At an (SNP) bifurcation two periodic branches, one stable and the other unstable, join together at a turning point. In the diagram there appears to be two turning points, but there is only one since the upper curve represents the maximum of the limit cycles and the lower curve represents the minimum.

In terms of the bifurcation diagram, a periodic branch has a turning point at an SNP bifurcation. One incoming periodic branch is stable, while the other is unstable.

There are several other types of periodic bifurcations, but we will only discuss a homoclinic bifurcation. This bifurcation terminates a periodic solution branch and the oscillation period approaches infinity as the bifurcation is approached (i.e., as $\mu \rightarrow \mu_c$). Beyond the bifurcation point the limit cycle no longer exists. We will discuss this more in the next chapter.

Both the SNP and homoclinic bifurcations are known as global bifurcations, since they cannot be studied through a local analysis of a steady state (as is done with the local bifurcations such as the Hopf or the stationary bifurcations). To study their stability one must form Poincaré sections and examine the eigenvalues of the linearization of the Poincaré map. These are called Floquet multipliers. An SNP bifurcation occurs when of these multipliers crosses through the unit complex circle at 1.