# Phase Oscillators

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## Phase Oscillators

Oscillations occur everywhere in biology, from the level of gene expression to the level of animal populations. This ubiquity motivated Art Winfree to describe an oscillator as simply as possible (1967). His work was further developed by others, most notably by Kuramoto (1976).

To set the stage, consider the system of equations

$$\dot{x} = \Lambda(r)x - \Omega(r)y \tag{1}$$

$$\dot{y} = \Omega(r)x + \Lambda(r)y \tag{2}$$

where $r^2 = x^2 + y^2$. Systems of this form are called lambda-omega systems. They are special, since if we change to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, the ODEs can be written as

$$\dot{r} = r\Lambda(r) \tag{3}$$

$$\dot{\theta} = \Omega(r) \quad . \tag{4}$$

This system has a circular limit cycle at any radius $r > 0$ for which $\Lambda(r^*) = 0$, since $\dot{r} = 0$ in this case. Is this a stable limit cycle?

$$\frac{d}{dr} r\Lambda(r) = \Lambda(r) + r\Lambda' \tag{5}$$

and at $r^*$,

$$\Lambda(r^*) + r^*\Lambda'(r^*) = 0 + r^*\Lambda'(r^*) \quad . \tag{6}$$

Hence, the limit cycle at $r = r^*$ is stable if and only if $\Lambda'(r^*) < 0$.

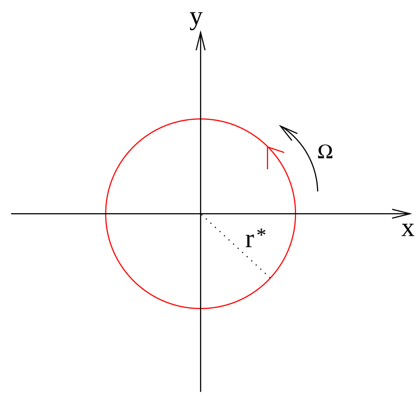



Figure 1: Circular limit cycle in the lambda-omega system.

The periodic solution travels around the circle (Fig. 1) with **angular velocity** of $\Omega(r^*)$. If the limit cycle is stable, then starting from any initial conditions (except the origin), the solution will eventually settle onto a regular oscillation with fixed amplitude and period $\frac{2\pi}{\Omega(r^*)}$. Hence, in the limit as $t \to \infty$ the system is described completely by its angular velocity around a circle. In this limit the system is called a phase oscillator, and it is a reduction of the planar system to a 1-dimensional flow on a circle.

## Entrainment of Fireflies

The following example comes from Strogatz (1994). In one species of southeast Asian fireflies the males gather in trees at night and begin flashing. Different flies flash at different frequencies when in isolation, and if they did this as a group then flashing in the trees would be continuous. However, periodic flashing is what is actually observed. This indicates that the fireflies synchronize their activity. This observation motivated experiments in the 1970's in which a flashlight was used to entrain flies to the flashing period of the flashlight.

Represent the flashing of the flashlight as a uniform phase oscillator:

$$\dot{\psi} = \Omega \tag{7}$$

where $\Omega$ is a constant. That is, let the flashlight turn on very briefly with some periodicity. Then represent this on a circle using

$$\text{Period} = \frac{2\pi}{\Omega} \tag{8}$$

or

$$\Omega = \frac{2\pi}{\text{period}} \tag{9}$$

and using Eq. 7 and replacing differentials by small changes $\Delta$,

$$\Delta\psi = \frac{2\pi\Delta t}{\text{period}} \quad . \tag{10}$$

So for a small time step $\Delta t$ the phase point moves counter-clockwise along the unit circle an angle $\Delta\psi$, as in Fig. 2.

Next, represent the firefly flashing as a nonuniform phase oscillator:

$$\dot{\theta} = \omega + A\sin(\psi - \theta) \tag{11}$$

where $\omega$ is the natural frequency of the firefly and $A$ is the **coupling strength** (assume that $A > 0$). If $\psi > \theta$, then $A\sin(\psi - \theta) > 0$ and the firefly speeds up, trying to catch up to the flashlight.

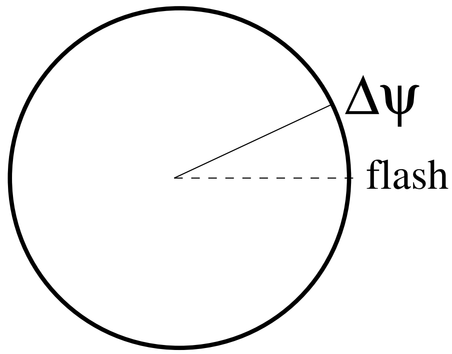



Figure 2: Unit circle representation of flashlight flashing.

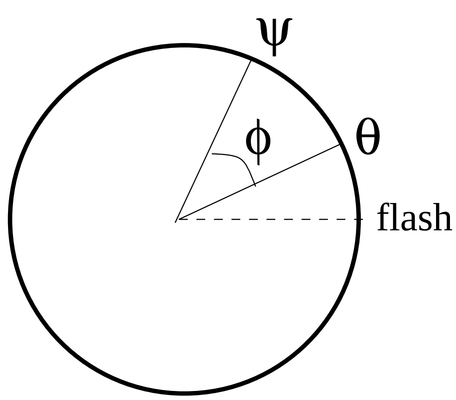



Figure 3: Phase oscillator representations of firefly ($\theta$) and flashlight ($\psi$). A flash occurs at a phase of 0.

Define the **phase difference** as $\phi \equiv \psi - \theta$ (Fig. 3). Then

$$\begin{aligned} \dot{\phi} &= \dot{\psi} - \dot{\theta} && (12) \\ &= \Omega - \omega - A \sin\phi \ . && (13) \end{aligned}$$

The 3 parameters ($\Omega$, $\omega$, and $A$) can be replaced by a single parameter $\mu$ by **non-dimensionalizing** the system:

$t$ has units of seconds

$\Omega$, $\omega$, and $A$ have units of radians per second.

Define the dimensionless variables $\tau \equiv At$ and $\mu \equiv \frac{\Omega - \omega}{A}$. Then

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{d\phi}{d\tau}\frac{d\tau}{dt} && (14) \\ &= \frac{d\phi}{d\tau} \cdot A && (15) \end{aligned}$$

and

$$\Omega - \omega - A\sin\phi = A\mu - A\sin\phi \ . \tag{16}$$

Combining, we get the phase equation in **dimensionless form**:

$$\frac{d\phi}{d\tau} = \mu - \sin\phi \tag{17}$$

## Parameter Exploration

In parametric systems it is typical to explore parameter space to get a better understanding of the range of behaviors that the system of equations can exhibit. In our dimensionless equation the parameter space is one-dimensional (unlike the 3-dimensional space of the original dimensional phase equation).

Case 1: $\mu = 0$

Here the natural frequency of the firefly is equal to the frequency of the flashlight. The phase difference equation is

$$\frac{d\phi}{d\tau} = -\sin\phi \tag{18}$$

Steady states: $\phi^* = 0, \pi$ (and mutliples of $2\pi$).

We can view the phase portraits of the phase difference on either the number line or the circle diagram.

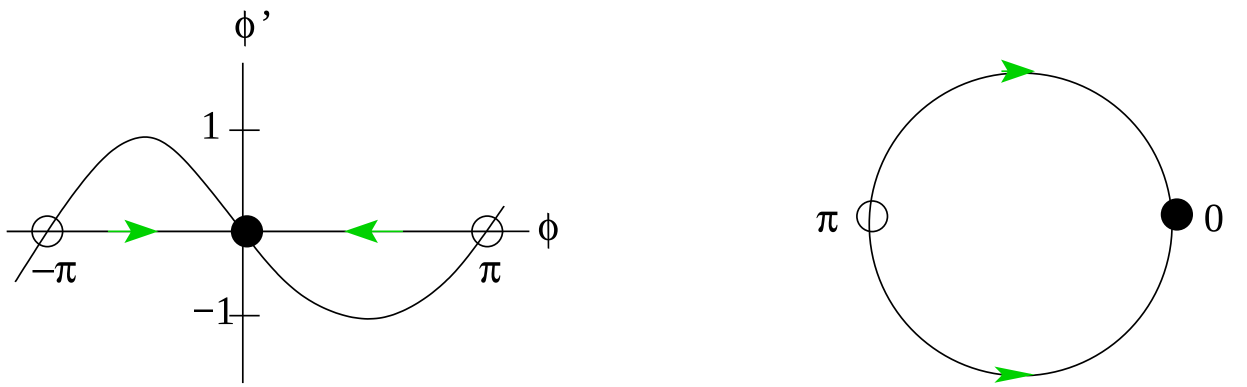



Figure 4: Dynamics of the phase difference, showing either the velocity function (left) or the dynamics on the unit circle (right).

From the velocity function, we see that $\phi^* = 0$ is stable. This is the in-phase solution, where the fly flashes at the same time as the flashlight.

Since the asymptotic phase difference is stable, the firefly is said to be entrained by the flashlight. Even if the phase of the firefly is initially different from that of the flashlight, it will eventually match up. Note that the flashlight is not effected by the firefly, so it does the entraining.

Case 2: $0 < \mu < 1$

Here the flashlight frequency is greater than the natural frequency of the firefly. When $\mu$ is increased beyond 0 it translates the velocity curve upward. The new stable steady state $\phi_1^*$ is at a non-zero value, so the firefly is still phase locked to the flashlight, but no longer in phase. Since the stable phase difference is positive, the flashlight flashes before the fly during each cycle.

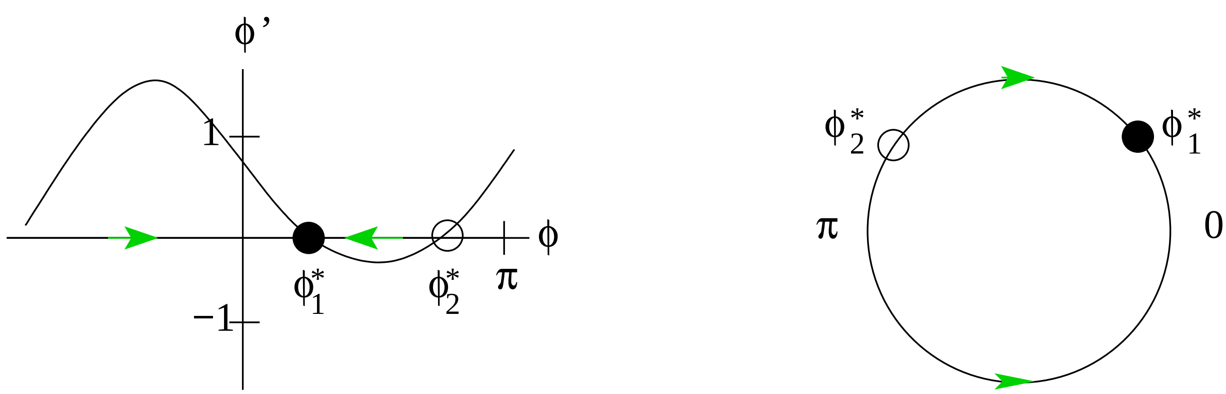



Figure 5: When $0 < \mu < 1$ the stable solution is not in phase. The flashlight leads the fly.

The two steady states (stable and unstable) satisfy

$$\phi^* = \arcsin(\mu) \tag{19}$$

and the smaller of the two solutions is the stable one. In terms of the phase circle diagram (Fig. 6) the phase of the flashlight is always greater than the phase of the firefly at steady state.

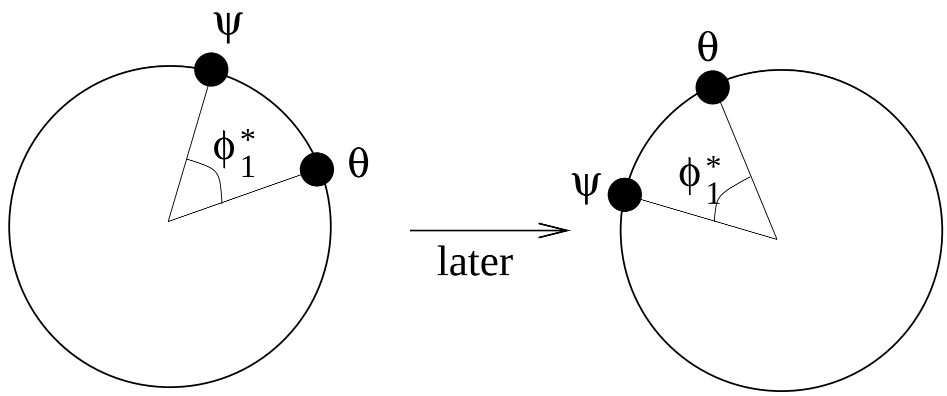



Figure 6: The flashlight leads the firefly by a phase angle of $\phi_1^*$

Case 3: $\mu = 1$

Here the difference between the flashlight frequency and the natural frequency of the firefly is equal to the amplitude of the coupling. In this case there is a single steady state,

$$\phi^* = \arcsin(1) = \frac{\pi}{2} \tag{20}$$

and the velocity curve is tangent to the $\phi$-axis. The system is at a saddle-node bifurcation (Fig. 7).

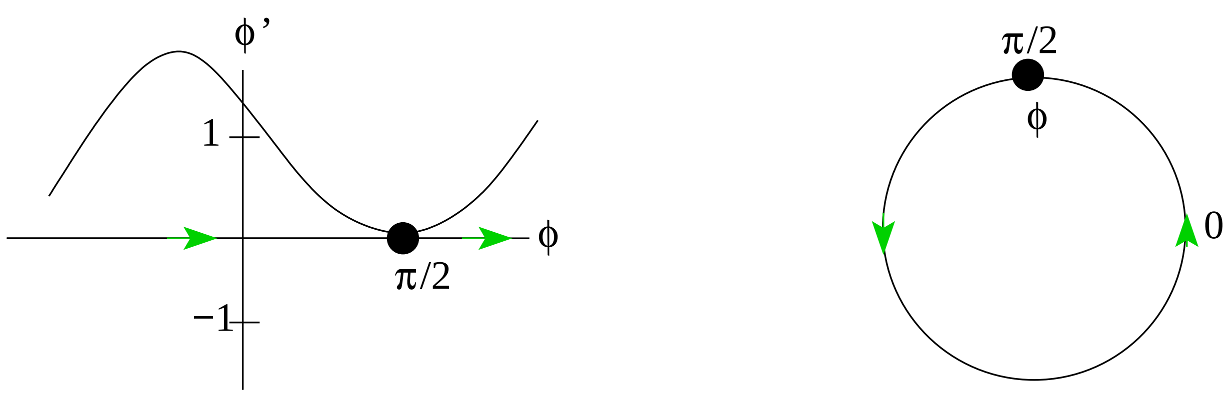



Figure 7: The equilibrium is half-stable at the saddle-node bifurcation

The steady state is **half stable** since it attracts on one side and repels on the other.

Case 4: $\mu > 1$

Here the difference between flashlight frequency and natural frequency of the firefly is greater than the coupling amplitude. Now the velocity curve never intersects the $\phi$-axis, so the phase difference continues to change over time. This is called phase drift (Fig. 8). In this case, the flashlight fails to entrain the firefly.

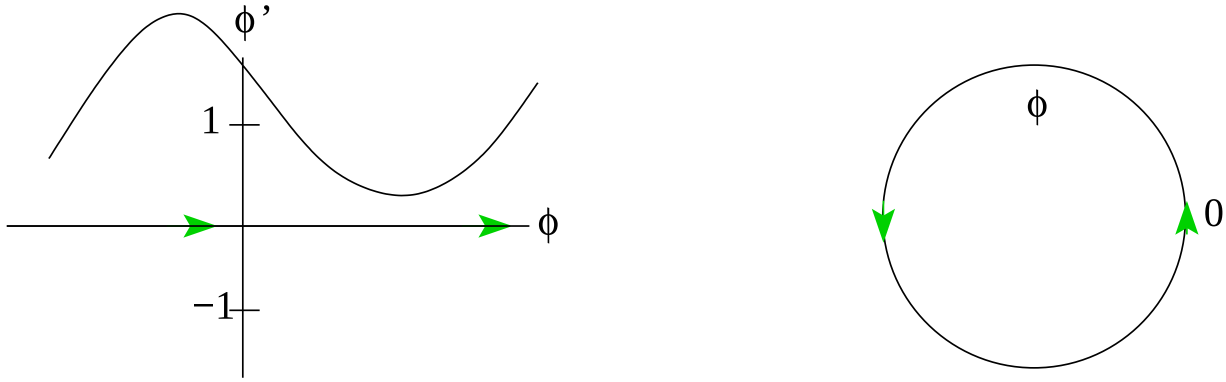



Figure 8: Phase drift occurs when $\mu > 1$.

If $\mu < 0$ the cases are similar to these, except that the firefly leads the flashlight ($\phi^* < 0$). We can then give the entrainment window, the range of $\mu$ over which the system is entrained. In terms of the dimensionless parameter $\mu$ this is shown in Fig. 9.

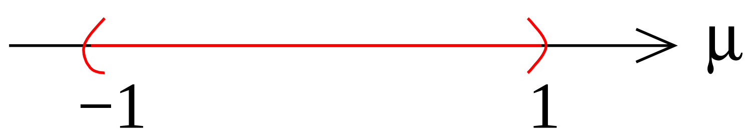



Figure 9: The entrainment window in terms of the dimensionless parameter $\mu$.

While in terms of the original parameters (dimensioned) it is shown in Fig. 10.

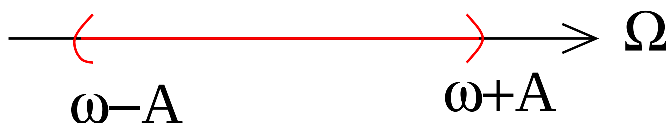



Figure 10: The entrainment window in terms of the dimensional parameter Ω.

## The $\theta$ Neuron Model

Bard Ermentrout developed a representation of neuron spiking that is based on a phase oscillator. This 1-dimensional model has an advantage over 1-dimensional leaky integrate-and-fire models in that it is a continuous dynamical system.

Recall that a type 1 oscillator has the bifurcation diagram shown below. The periodic branch ends in a homoclinic bifurcation, so it has infinite pe-

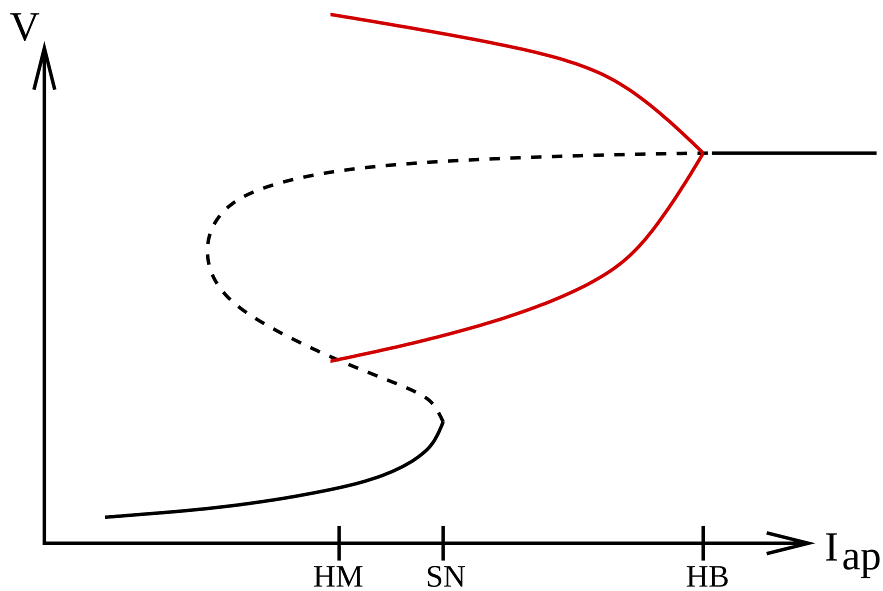



Figure 11: black=stationary branch, red=periodic branch, HB=Hopf bifurcation, SN=saddle-node bifurcation, HM=saddle-loop homoclinic bifurcation

riod. With an appropriate adjustment of parameter values, the homoclinic can be moved to the lower saddle-node bifurcation, producing a saddle-node on an invariant circle (SNIC) bifurcation. The $\theta$ neuron model has a bifurcation diagram of this form. In fact, it was created with this bifur-

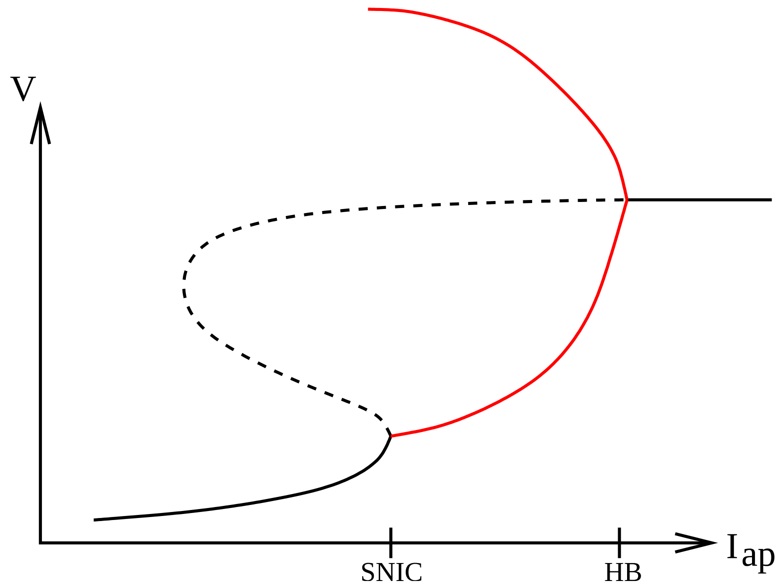



Figure 12: Type-1 oscillator with a SNIC bifurcation.

cation diagram in mind.

The normal form for a saddle-node bifurcation is

$$\frac{dx}{dt} = qx^2 + pI \tag{21}$$

where $q > 0$, $p > 0$, and $I$ are parameters. Note that at equilibrium,

$$x = \pm\sqrt{\frac{-pI}{q}} \tag{22}$$

so the number of equilibria depends on $I$. There is clearly a saddle-node bifurcation at $I = 0$.

The ODE in Eq. 21 is problematic in that $x \to \infty$ when $I > 0$. To fix this, we use a change of variables,

$$\boxed{x = \tan\left(\frac{\theta}{2}\right)} \tag{23}$$

With this change of variables, $\theta \to \pi$ as $x \to \infty$.

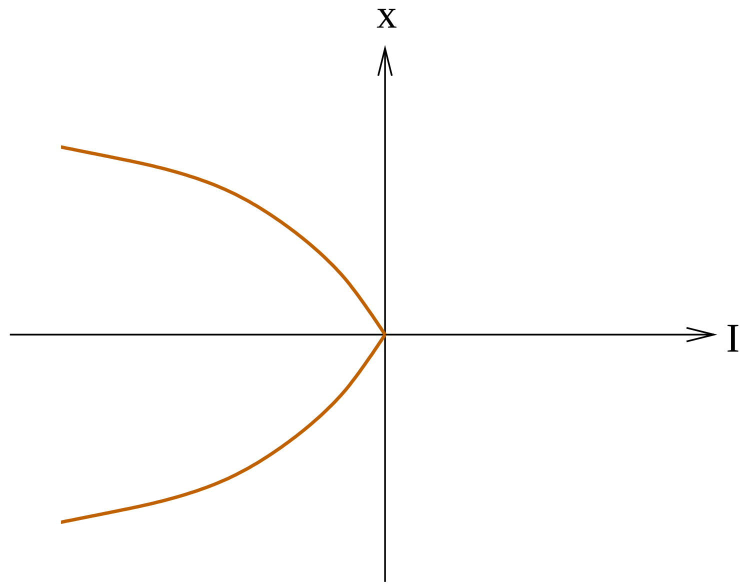



Figure 13: Graph of Eq. 22. The curve represents stationary solutions without an indication of stability.

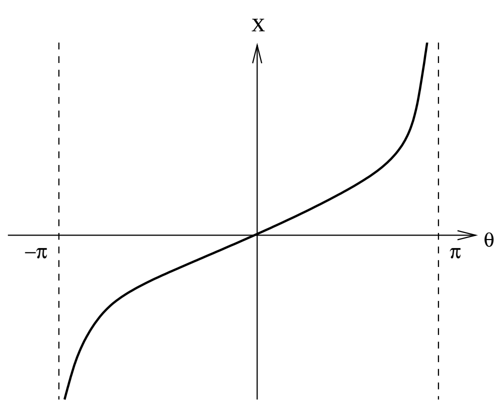



Figure 14: Graph of Eq. 23

Differentiating,

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\tan\left(\frac{\theta}{2}\right) \\
&= \sec^2\left(\frac{\theta}{2}\right)\frac{d}{dt}\frac{\theta}{2} \\
&= \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{d\theta}{dt} \\
&= \frac{1}{2\cos^2(\frac{\theta}{2})}\frac{d\theta}{dt} \\
&= \frac{1}{1+\cos\theta}\frac{d\theta}{dt} \quad .
\end{aligned} \tag{24}
$$

Also,

$$
\begin{aligned}
qx^2 &= q\tan^2\left(\frac{\theta}{2}\right) \\
&= q\left(\frac{1-\cos\theta}{1+\cos\theta}\right)
\end{aligned} \tag{25}
$$

so Eq. 21 becomes

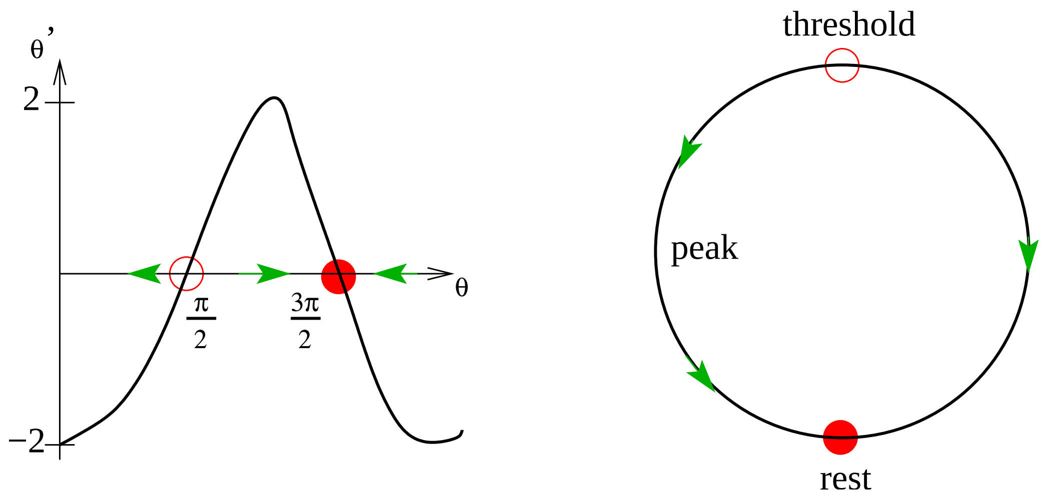
$$
\frac{1}{1+\cos\theta}\frac{d\theta}{dt} = q\left(\frac{1-\cos\theta}{1+\cos\theta}\right) + pI \tag{26}
$$

or

$$
\boxed{\frac{d\theta}{dt} = q(1-\cos\theta) + pI(1+\cos\theta)} \tag{27}
$$

This is the $\theta$ Model.

The right hand side of Eq. 27 is periodic with period of $2\pi$, so one can visualize the dynamics on a unit circle. Let's look at several cases, all with $p = q = 1$ and with different values of $I$.




<u>$I = -1$</u>

The ODE is

$$\frac{d\theta}{dt} = -2\cos\theta \quad . \tag{28}$$

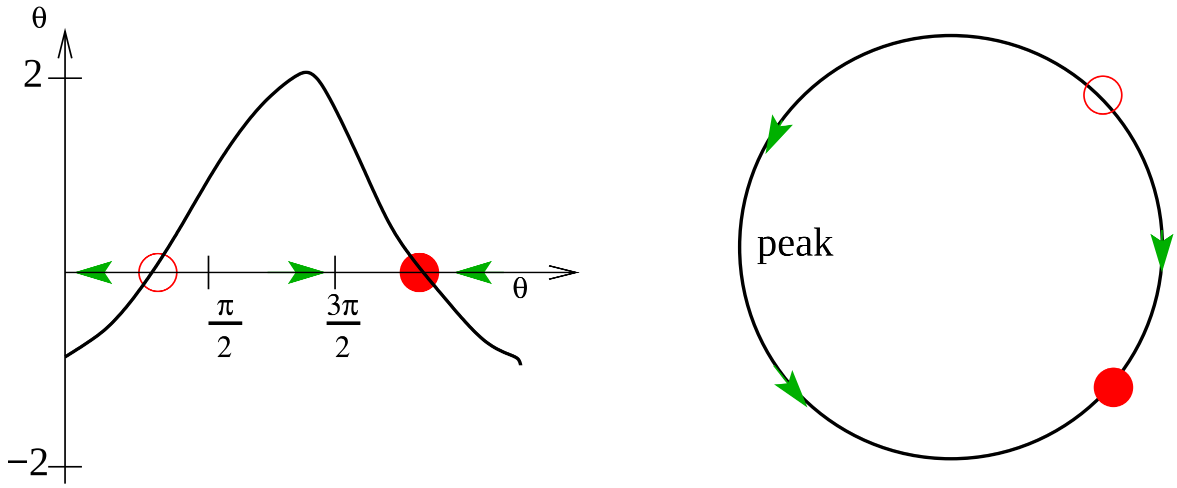
The rest state is at $\theta = \frac{3\pi}{2}$ and the spike threshold is at $\theta = \frac{\pi}{2}$. The locations of the rest state and the spike threshold are far apart, due to the large applied hyperpolarizing (i.e., negative) current. If the system is at rest, but perturbed past the spike threshold then a spike is produced. This is best viewed on the unit circle.

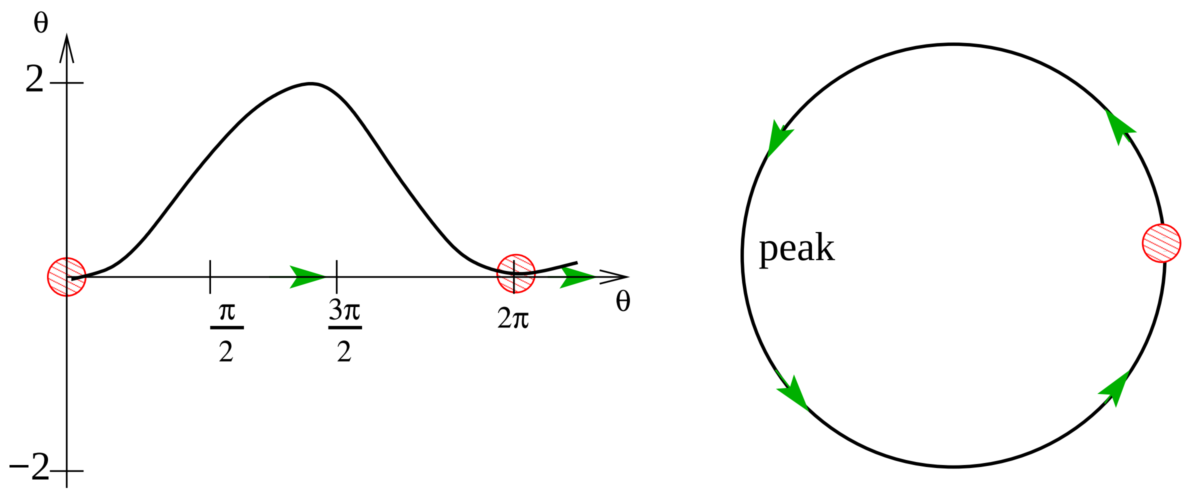
<u>$I = -\frac{1}{2}$</u>

Because there is less hyperpolarizing applied current, the spike threshold is closer to the rest state. The model cell is now more excitable.

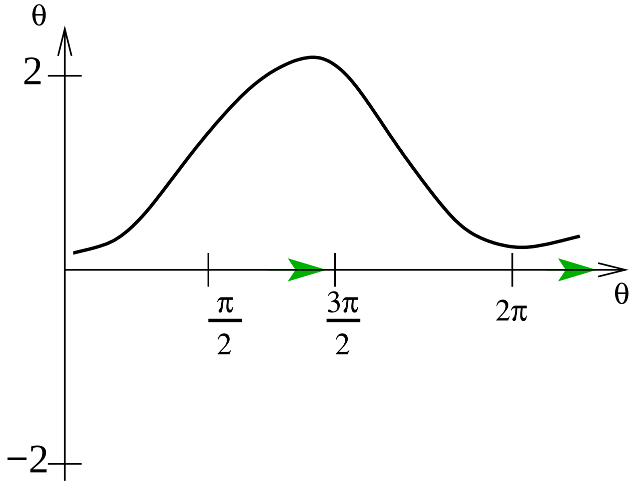
<u>$I = 0$</u>

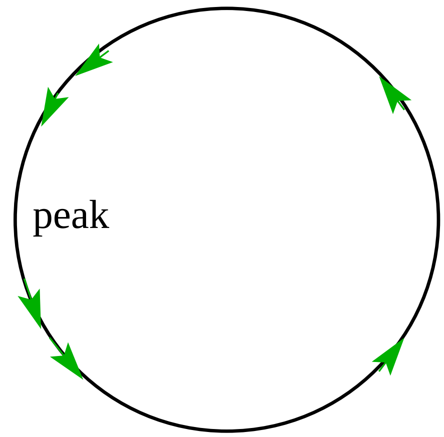
With no applied current the rest state and threshold have coalesced, forming a half-stable steady state. This is a **saddle-node bifurcation**.


θ
2
−2
π/2
3π/2
θ
peak


θ
2
−2
π/2
3π/2
2π
θ
peak

$I = \epsilon$, $\epsilon > 0$ small

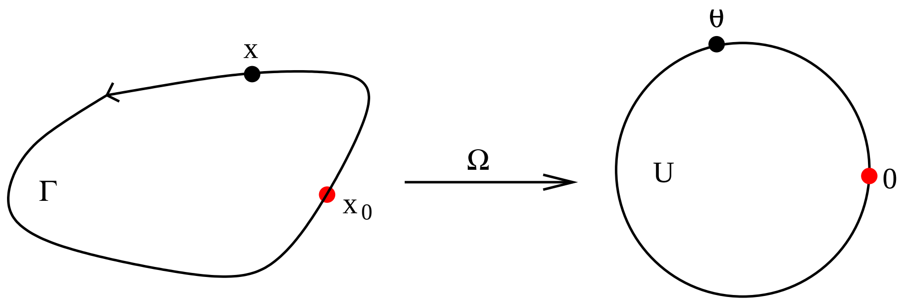



Now there is periodic spiking. As $\epsilon \to 0^+$ the spike period approaches $\infty$ due to the bottleneck near $\theta = 0$. Thus, there is a homoclinic bifurcation at $I = 0$, in addition to the sadle-node bifurcation at $I = 0$. This is the saddle-node on invariant circle (SNIC) bifurcation that we wanted.

## Isochrons

Recall that the idea behind a phase oscillator is that we map each point on a limit cycle to a point, or phase, on the unit circle.




Let $\Gamma$ denote a stable limit cycle for some system of differential equations. Then there exists a continuous function $\Omega$ that maps each point on $\Gamma$ to a point on the unit circle $U$. That is, one can pick a point on $\Gamma$, call it $x_0$, and map it to $\theta = 0$. Then every other point on $\Gamma$ is mapped to another point $\theta$ on $U$ in a continuous fashion.

The dynamics of the original stable limit cycle are replicated by the phase oscillator

$$\frac{d\theta}{dt} = \Omega \tag{29}$$

$\theta$ is the **phase** relative to $x_0$ and $\Omega$ is the **phase speed**.

If the limit cycle is stable, then we can extend this idea to all points within the basin of attraction of the limit cycle. We do this through the use of isochrons. An isochron is a set of points that all have the same **asymptotic phase**. That is, all points on an isochron will approach the same phase on the limit cycle as $t \to \infty$. There will be an isochron for each phase value, and thus each point in the basin of attraction will lie on an isochron. In this way, we define the phase of each point in the basin according to which isochron it lies on.

Example (Winfree, 2001)

Consider the polar system of equations:

$$\frac{dr}{dt} = (1-r)r^2 \tag{30}$$

$$\frac{d\theta}{dt} = r \quad . \tag{31}$$

There is a stable solution of $r = 1$, which is a circular limit cycle. The basin of attraction is the entire punctured plane, with the point at the origin removed. For initial conditions with $r < 1$ the phase point moves outward in a counterclockwise direction to the unit circle. For initial conditions with $r > 1$ the motion is inward toward the unit circle. How can one compute the isochrons? It's not easy, and the technique differs for different problems. In this case, on the limit cycle the asymptotic phase is just $\theta$. Thus, we can define the phase of a point, $\phi(r,\theta)$, in the basin at location $(r,\theta)$ as:

$$\phi(r,\theta) = \theta - f(r) \tag{32}$$

for some function $f(r)$ satisfying $f(1) = 0$. We want the curve where

$\phi = C$, a constant. That is,

$$C = \theta - f(r) \tag{33}$$

For a point on the limit cycle ($r = 1$) with phase angle $C$ we have

$$\frac{d\theta}{dt}\,|_{r=1} = 1 \tag{34}$$

from Eq. 31. For all points on the isochron through $C$, the asymptotic phase angle is the same as that of $C$. That is,

$$\frac{d\phi}{dt} = \frac{d\theta}{dt}\,|_{r=1} = 1 \tag{35}$$

for all points on this isochron. Thus,

$$\frac{d\phi}{dt} = 1 = \frac{d\theta}{dt} - \frac{df}{dr}\frac{dr}{dt} \tag{36}$$

or

$$1 = r - \frac{df}{dr}(1-r)r^2 \tag{37}$$

so that

$$\frac{df}{dr} = \frac{1-r}{-(1-r)r^2} = -\frac{1}{r^2} \quad . \tag{38}$$

Integrating,

$$f(r) = \frac{1}{r} + constant \quad . \tag{39}$$

But on the limit cycle ($r = 1$) we want $f = 0$, so we pick $constant = -1$. Thus, $f(r) = \frac{1}{r} - 1$. Therefore, the isochron with phase C is

$$\theta = C + \frac{1}{r} - 1 \tag{40}$$

and for general phase $\phi$,

$$\theta = \phi + \frac{1}{r} - 1 \ . \tag{41}$$

This can be rewritten as

$$\phi = \theta + 1 - \frac{1}{r} \tag{42}$$

where $\phi$ is the phase of a point at initial location $(r, \theta)$. We see from this that $\phi > \theta$ when $r > 1$ (outside of limit cycle) and $\phi < \theta$ when $r < 1$ (inside of limit cycle). This is illustrated in Fig. 15 below, and four isochrons and the limit cycle are shown in the next figure. These isochrons are graphs of

$$r = \frac{1}{\theta + 1 - \phi} \tag{43}$$

for several values of $\phi$.

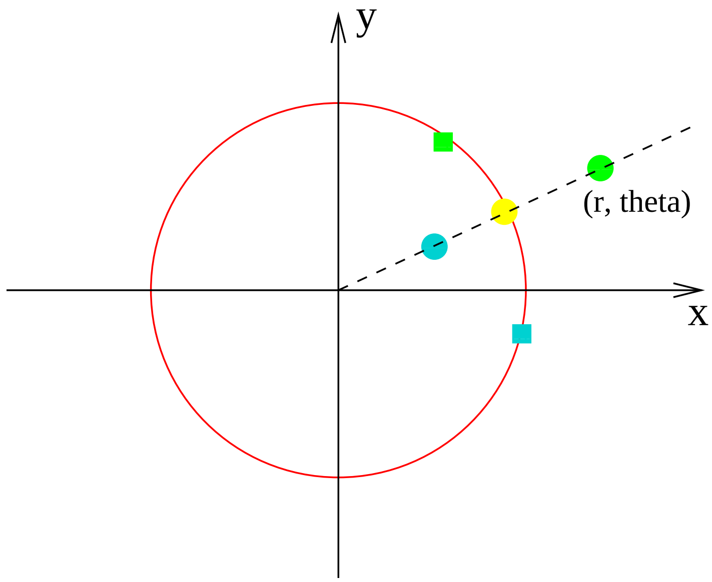
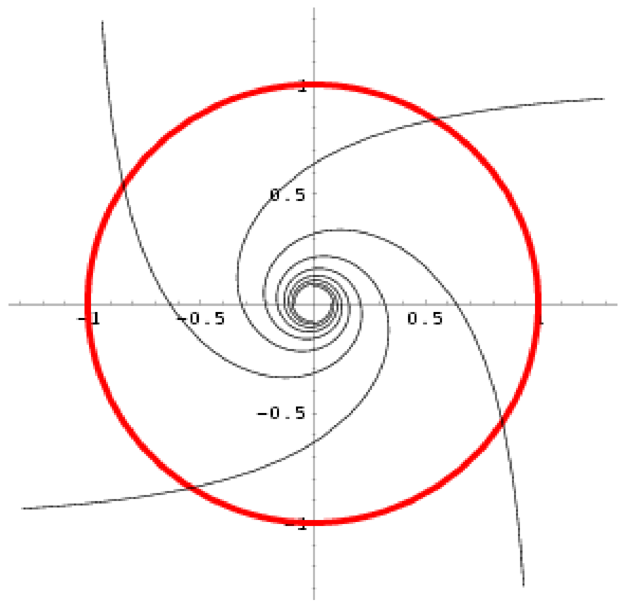



Figure 15: Points outside of the limit cycle (green circle) have asymptotic phase (green square) greater than $\theta$ (yellow circle). Points inside the limit cycle (cyan circle) have asymptotic phase (cyan square) less than $\theta$.

Isochrons are useful for understanding phase transition curves and phase response curves. Suppose that a phase point moving along a limit cycle is given an instantaneous perturbation, knocking it off the limit cycle but not out of the basin of attraction of the limit cycle. Then the point will land on an isochron, which tells us the phase of the point once it comes back to the limit cycle. The phase transition curve (PTC) plots the new asymptotic phase of the point as a function of the stimulus phase. The phase response curve (PRC) plots the phase difference as a function of the stimulus phase.

Example

Consider a planar system with the following circular stable limit cycle and



1
0.5
-1
-0.5
0.5
1
-0.5
-1

radial isochrons:

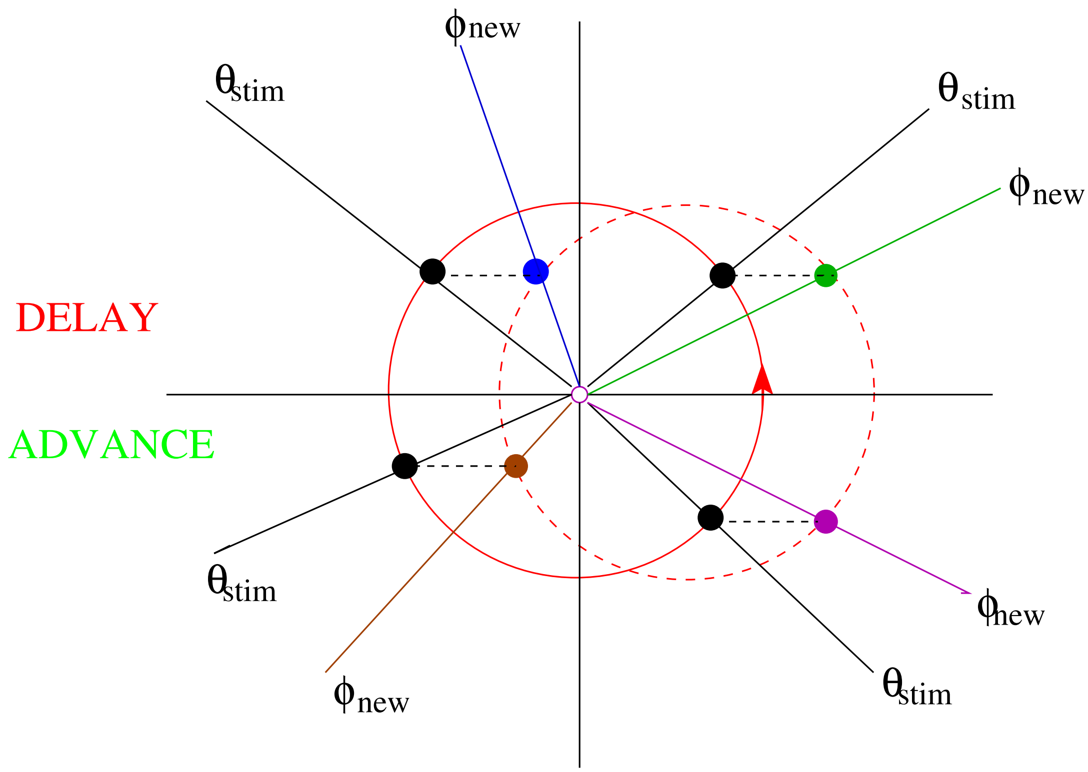



Figure 16: Limit cycle, isochrons, and image for weak perturbations.

The solid circle is the limit cycle. The dashed circle is the result of a small horizontal perturbation applied to any point on the limit cycle. Consider the case illustrated with a black horizontal line. The isochron intersecting the solid circle defines the original phase, while the isochron intersecting the dashed circle defines the new phase. The phase transition curve for this example is shown below.

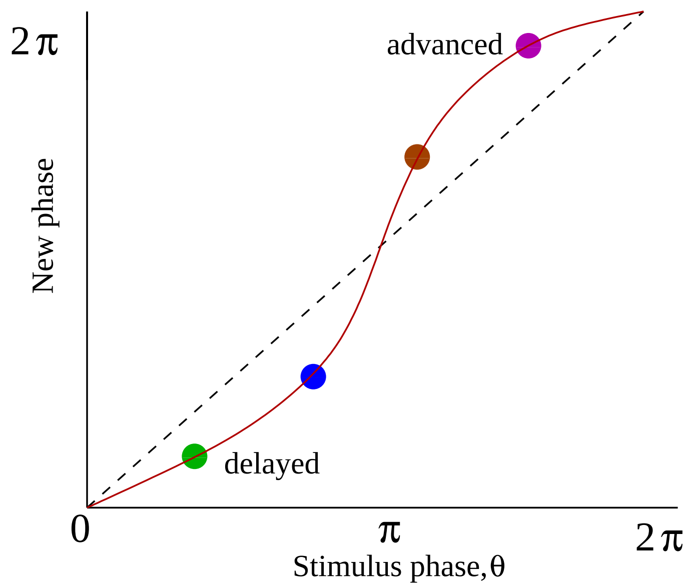



Figure 17: Type 1 Phase Transition Curve (PTC) for weak perturbations

For values of $\theta$ at which the PTC is below the dashed $45^o$ line the phase is **delayed** by the perturbation, while for values at which the PTC is above the line the phase is **advanced**.

The **phase response curve (PRC)** is formed by computing the difference between the PTC and the dashed line (no phase shift) at each $\theta$, as shown below.

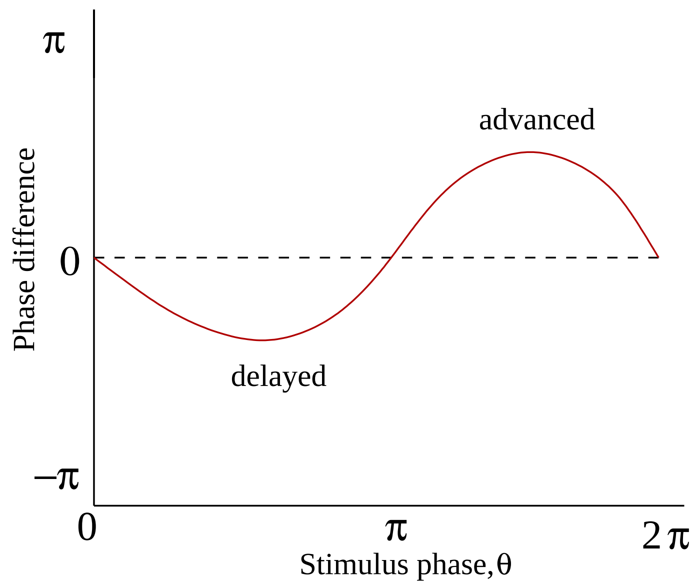



Figure 18: Type 1 Phase Response Curve (PRC) for weak perturbations

In this last example the perturbations were weak, and the resetting response is an example of Type 1 phase resetting. For large perturbations the situation is quite different:

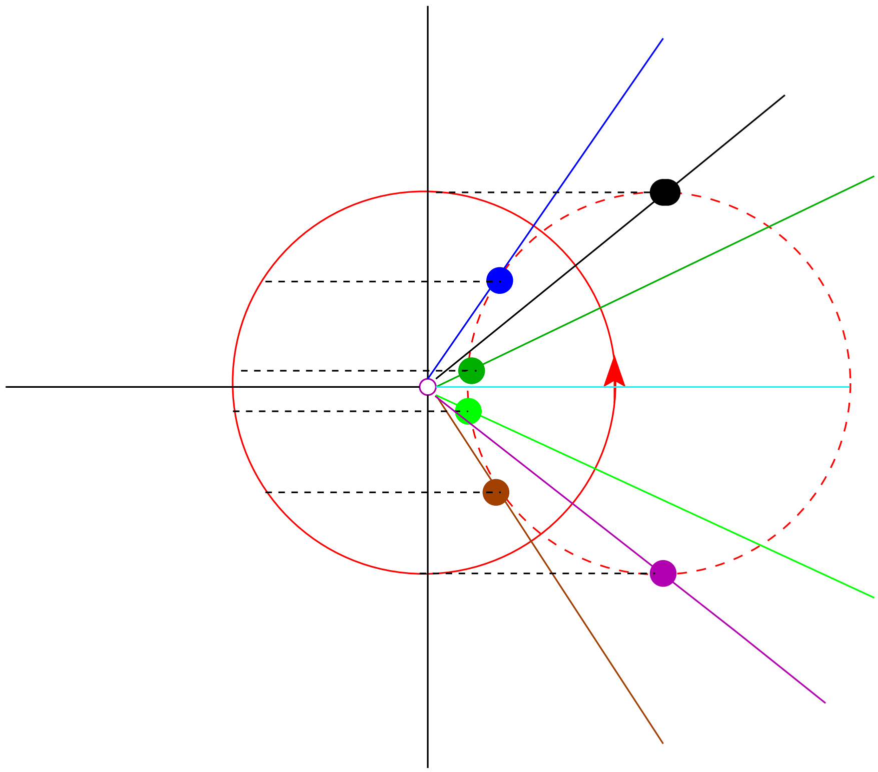


Figure 19: Limit cycle, isochrons, and image for strong perturbations.

When a perturbation is applied at $\theta \in (0, \pi)$ the response is delayed, just as before. However, while the new phase was previously a monotonic increasing function of the old phase, it now first increases but then decreases on the interval $\theta \in (0, \pi)$. The blue isochron corresponds to the maximal new phase value. At $\theta = \pi$ the new phase is 0 (light blue isochron) and for larger $\theta$ becomes negative. The new phase grows in magnitude initially, but then at some point (brown isochron) begins to decline toward 0. This is called Type 0 resetting.

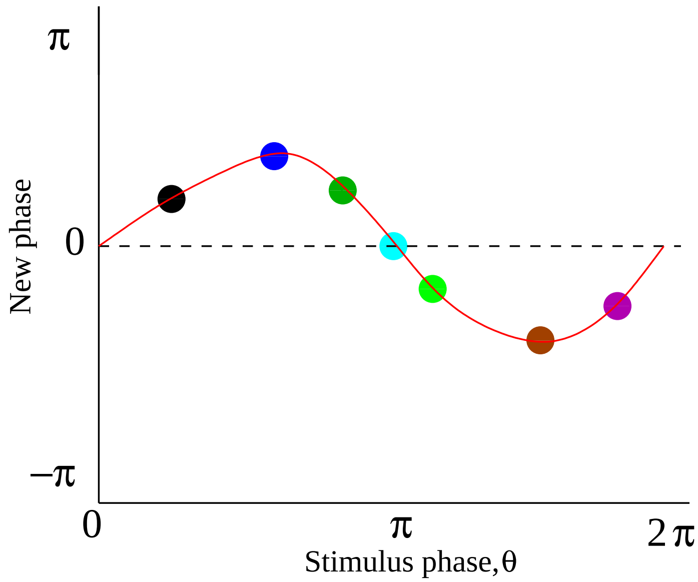



Figure 20: Type 0 PTC for strong perturbations

If one restricts the phase to the interval $[0, 2\pi)$, then the Type 0 PTC can be represented as follows:

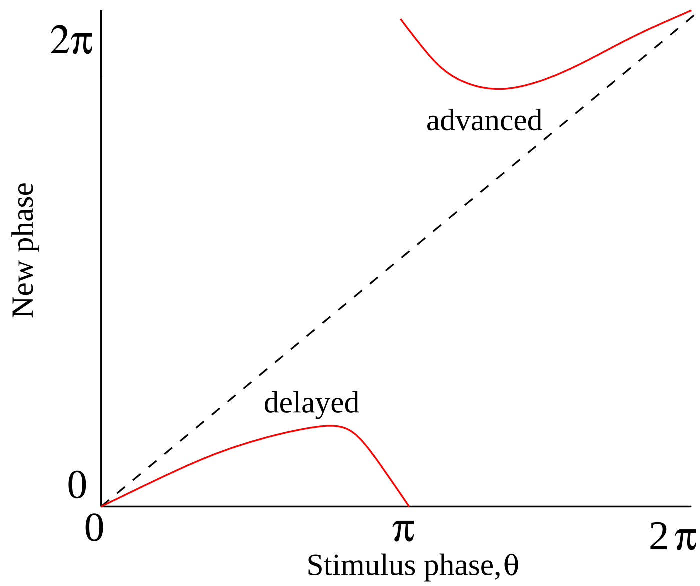



Figure 21: Alternate form of Type 0 PTC

What does the Type 0 PRC look like?

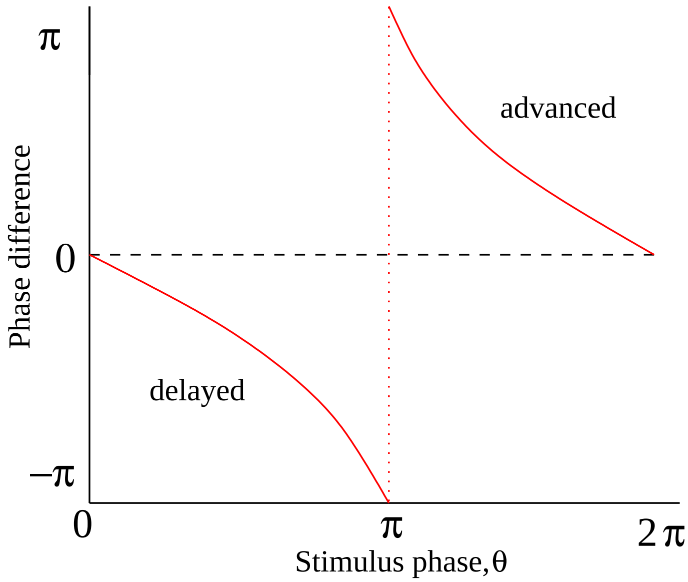



Figure 22: Type 0 PRC for strong perturbations

The type 1 and type 0 PRCs are fundamentally different. In the first case the curve is continuous, while in the second case it has a discontinuity. How did the type 0/1 terminology come about? This is based on the winding number. Note that in the case of weak perturbations the "shifted limit cycle" (after perturbation) contained the steady state at the origin and it therefore crossed all of the isochrons. Thus, any new phase (associated with an isochron) could be achieved through a perturbation given at the right time. This is why the phase transition curve is continuous. The winding number is defined as the number of times that the shifted limit cycle traverses a complete cycle as defined by the isochrons of the original cycle. For the case of weak perturbations in this example, the winding number is 1, and for this reason the PTC curve is called "type 1". Next consider the case of strong perturbations, where the new limit cycle (after perturbation) is shifted past the steady state. Now there is a whole family of isochrons that do not cross the shifted limit cycle. For example, none of the isochrons between $\pi/2$ and $3\pi/2$ cross the shifted limit cycle. Therefore, there is no perturbation of the size given here that will bring the system to a new phase between $\pi/2$ and $3\pi/2$. Since the shifted limit cycle does not traverse a complete cycle of phases the winding number is 0. Thus, the PTC is a "type 0" PTC.

Phase transition and response curves are important both in understanding synchronization properties of weakly coupled oscillators and for understanding something about an actual biological oscillator. In the latter case, it is often possible to determine PTCs and PRCs experimentally. These give insight into how sensitive the system is to perturbations, and place constraints on mathematical models of the system (i.e., the model PRC should look similar to the experimental one).

## Synchronization of Weakly Coupled Oscillators

Here we see how the PRC can help in the analysis of two endogenous oscillators that are mutually coupled. The mathematical theory is most developed for the case where coupling is weak, since then the oscillators can be treated as phase oscillators.

Consider the two coupled oscillators

$$\frac{d\vec{x}_1}{dt} = f_1(\vec{x}_1) + \vec{g}_{12}(\vec{x}_1, \vec{x}_2)$$
$$\frac{d\vec{x}_2}{dt} = f_2(\vec{x}_2) + \vec{g}_{21}(\vec{x}_1, \vec{x}_2) \tag{44}$$

where $f$ is the oscillator's velocity function and $g$ is the coupling function. Also, $\vec{x} \in \Re^n$ where $n \geq 2$. The corresponding phase model is

$$\dot{\theta}_1 = \omega_1 + Q_1(\theta_1) g_{12}(\theta_1, \theta_2)$$
$$\dot{\theta}_2 = \omega_2 + Q_2(\theta_2) g_{21}(\theta_1, \theta_2) \tag{45}$$

where $\omega$ is a natural frequency of the oscillator and $Q(\theta)$ is the infinitesimal PRC. Here, $g_{12}$ represents the strength of perturbation of oscillator 2 onto oscillator 1 as a function of the two phases, and $Q_1$ represents the effect that a small perturbation has on oscillator 1 as a function of its phase.

The infinitesimal PRC $Q$ is defined as the PRC normalized by the amplitude of the pertubation, $A$, in the limit as $A \to 0$:

$$Q(\theta) = \lim_{A \to 0} \frac{\text{PRC}}{A} \quad . \tag{46}$$

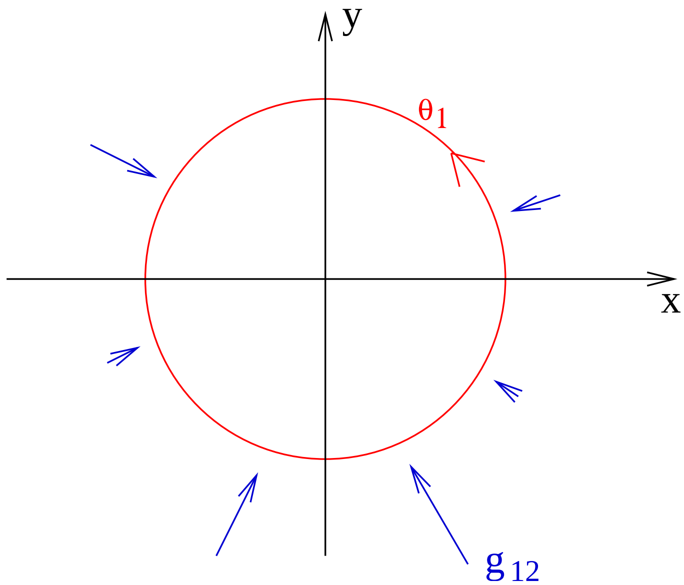



Figure 23: The magnitude of the perturbing input to $\theta_1$ depends in general on the phase of each oscillator.

Equation 45 can be rewritten as

$$\dot{\theta}_1 = \omega_1 + h_{12}(\theta_1, \theta_2)$$
$$\dot{\theta}_2 = \omega_2 + h_{21}(\theta_1, \theta_2) \tag{47}$$

where $h_{12} = Q_1 g_{12}$ describes the influence of oscillator 2 on oscillator 1.

One can then average the influence of one oscillator on the other by averaging $h$ over all interactions that occur at a given phase difference. Let $\chi = \theta_2 - \theta_1$, so that $\theta_2 = \chi + \theta_1$. Then

$$h_{12}(\theta_1, \theta_2) = h_{12}(\theta_1, \theta_1 + \chi) \ . \tag{48}$$

We can then compute the H-function:

$$\boxed{H_{12}(\chi) = \frac{1}{2\pi} \int_0^{2\pi} h_{12}(\theta_1, \theta_1 + \chi)\, d\theta_1} \tag{49}$$

also,

$$H_{21}(-\chi) = \frac{1}{2\pi} \int_0^{2\pi} h_{21}(\theta_2 - \chi, \theta_2)\, d\theta_2 \quad . \tag{50}$$

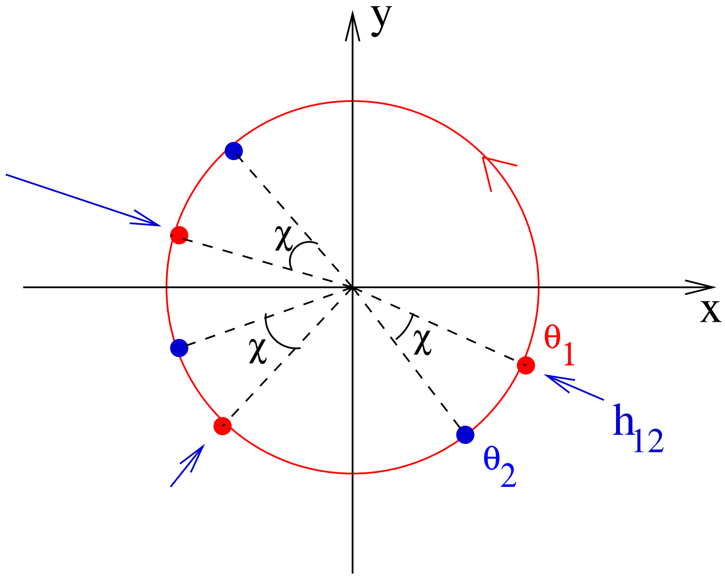



Figure 24: The H-function averages the coupling input over $\theta_1$ for a fixed phase difference $\chi$.

Then the phase equations become

$$\dot{\theta}_1 = \omega_1 + H_{12}(\chi) \tag{51}$$

$$\dot{\theta}_2 = \omega_2 + H_{21}(-\chi) \tag{52}$$

or

$$\boxed{\dot{\chi} = \omega + H(\chi)} \tag{53}$$

where $\omega \equiv \omega_2 - \omega_1$ and $H(\chi) \equiv H_{21}(-\chi) - H_{12}(\chi)$. The system is now 1-dimensional!

**Example**

Consider two coupled oscillators with different natural frequencies, so $\omega \neq$

0. To determine a phase locked solution or synchronized solution ($\chi^*$) one finds the steady state solutions to this phase difference ODE. If there is no steady state, then the system will exhibit phase drift. The steady state solutions satisfy

$$\boxed{\omega + H(\chi^*) = 0} \ . \tag{54}$$

In the Kuramoto model $H(\chi) = -\sin(\chi)$. To find synchronized solutions in this case, one looks for solutions of $\omega - \sin(\chi) = 0$.

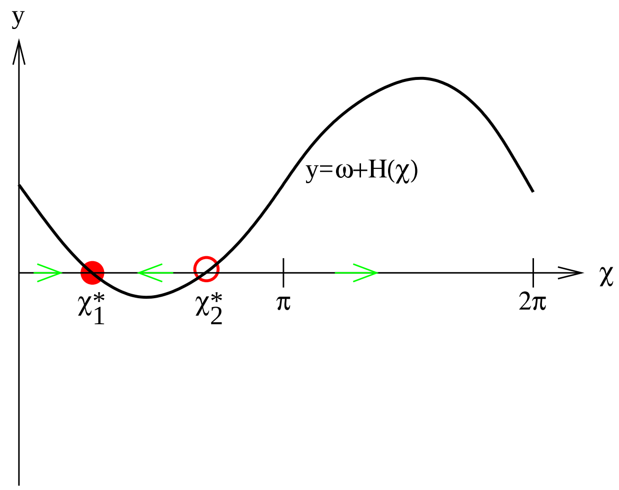



Figure 25: Two synchronized solutions, one stable and the other unstable. $0 < \omega < 1$.

In the stable phase-locked steady state oscillator 2 **leads** oscillator 1 by a phase angle of $\chi_1^*$, since $0 < \chi_1^* < \pi$. If $\chi_1^*$ had been greater than $\pi$, then oscillator 2 would **lag** oscillator 1.

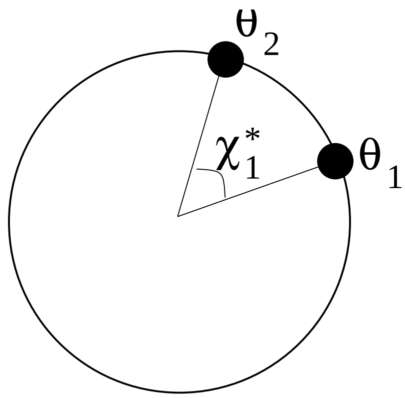



Figure 26: When synchronized, oscillator 2 leads oscillator 1 by a phase angle $\chi_1^*$ .

If the oscillators are actually identical, then $w = 0$ and the two synchronized solutions are at $\chi = 0$ (the **in-phase solution**) and $\chi = \pi$ (the **antiphase solution**). The in-phase solution is stable.

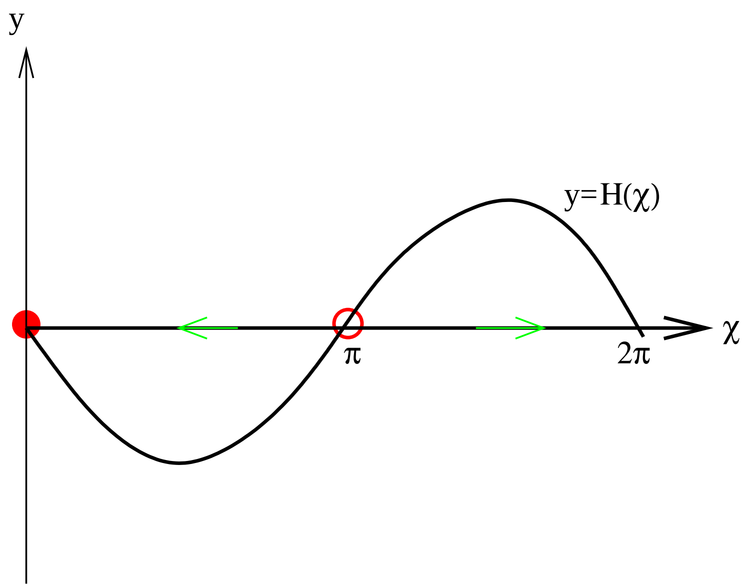



Figure 27: For identical oscillators the in-phase solution is stable. $\omega = 0$.

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