MAA 4227 ADVANCED CALCULUS II HOMEWORK 1

BOWERS

Theorem 1 (**Kummer's Test**)**.** *Let* $\sum a_n$ *be a positive series and* $\{p_n\}$ *a sequence of positive constants such that*

$$\lim_{n\to\infty}\Big[p_n\frac{a_n}{a_{n+1}} - p_{n+1}\Big] = L$$

exists, $-\infty \le L \le \infty$. *If* $0 < L \le \infty$, *then the series* $\sum a_n$ *converges. If* $-\infty \le L < 0$ *and if* $\sum \frac{1}{p_n}$ *diverges, then the series* $\sum a_n$ *diverges.*

Theorem 2 (**Raabe's Test**)**.** *Let* $\sum a_n$ *be a positive series and assume that*

$$\lim_{n\to\infty} n\Big(\frac{a_n}{a_{n+1}} - 1\Big) = L$$

exists, $-\infty \le L \le \infty$.
(i) If $1 < L \le \infty$, *then* $\sum a_n$ *converges;*
(ii) if $-\infty \le L < 1$, *then* $\sum a_n$ *diverges;*
(iii) if $L = 1$, *then* $\sum a_n$ *may either converge or diverge and the test fails.*

Theorem 3. *Let* $\sum a_n$ *be a positive series and assume that*

$$\lim_{n\to\infty} \log n\Big[n\Big(\frac{a_n}{a_{n+1}} - 1\Big) - 1\Big] = L$$

exists, $-\infty \le L \le \infty$.
(i) If $1 < L \le \infty$, *then* $\sum a_n$ *converges;*
(ii) if $-\infty \le L < 1$, *then* $\sum a_n$ *diverges;*
(iii) if $L = 1$, *then* $\sum a_n$ *may either converge or diverge and the test fails.*

Exercise 1. Item (i) is used in the proof of item (ii).
(i) Prove that

$$\lim_{n\to\infty}(n+1)\log\Big(\frac{n+1}{n}\Big) = 1.$$

You may assume that the logarithm function satisfies $\lim_{x\to a}\log x = \log a$ for each positive real number a.
(ii) Prove that Theorem 3 follows from applying Kummer's Test with $p_n = n\log n$.

Exercise 2. In each of (i) through (iii), decide for either convergence or divergence.

Date: Due: 11am–Monday–2–February–2004.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{1}{2n+1}$.

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots 2n}\cdot\frac{4n+3}{2n+2}$.

(iii) $\displaystyle\sum_{n=1}^{\infty} \left[\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdot 5\cdots(2n+1)}\right]^p$, where $p>0$.

Exercise 3. Define for each real number a and positive integer n the rising factorial $(a)_n$ by

$$(a)_n = a(a+1)\cdots(a+n-1).$$

For convenience, also define $(a)_0 = 1$. Notice that for each nonnegative integer n, $n! = (1)_n$. For real numbers a, b, c with c not zero or a negative integer, prove that the **hypergeometric series**

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(1)_n(c)_n}$$

converges if and only if $c-a-b>0$.

Exercise 4. Problem 16, page 81 of Rudin.

Exercise 5. Problem 17, page 81 of Rudin.