Optimal Impulse Control in the Foreign Exchange Market

Paper for PCE

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1 Introduction

For countries dependent on foreign trade and/or foreign capital, exchange rate policy is very important for which the Central Bank is responsible. The main objective of the Central Bank is usually to avoid the volatility of the exchange rate by keeping it as close as possible to a given target. To achieve this objective, the Central Bank may intervene in the foreign exchange market by purchasing or selling reserves. This intervention is costly: either the Central Bank has to accumulate reserves with a negative effect on the liquidity of the system or it will lose reserves and will be seen as unable to sustain the exchange rate in the future. To model this scenario, consider a currency with exchange rate dynamics modeled as a geometric brownian motion. In order to keep this exchange rate as close as possible to a given target, there is a running cost associated to the difference between the exchange rate and the target. Additionally, there are also fixed and proportional costs associated with each intervention. The objective is to minimize the total cost. This model is called the impulse stochastic control problem.

Another tool the Central Bank can use to influence the exchange rate is the level of the domestic interest rate. The higher the interest rate level, the more attractive domestic securities will be for international investors and the more valuable the domestic currency will be. This scenario is an application of the so-called instantaneous/classical stochastic control model.

Compared to the abundance of literature on instantaneous stochastic control [10] there are only a few books and papers on impulse stochastic control problems. The first person to apply the theory of impulse stochastic control to exchange rate is Jeanblanc-Picque [4], and later extended by Korn [5, 6]. Both considered a specified zone within which the exchange rate is to be contained, and found the optimal sizes of interventions required. In obtaining some facts for the impulse control problem, Korn [5, 6] made an analogous analysis to the theory of the classical stochastic control. In [5], he also gave an iterative method for finding the mimum of the intervention cost in the model described in the first paragraph.

Neither Jeanblanc-Picque nor Korn provided an exact analytic solution to the problem. Cadenillas and Zapatero [1] solved the problem by applying the theoy of stochastic impulse controls and for the first time, provided a numerical solution as well as some comparative statics analysis that yields interesting results about the effects of the changes of parameters on the optimal intervention strategy.

Cadenillas and Zapatero [1] didn't consider the effects that the reaction of the participants to the intervention of the Central Bank might have on the dynamics of the exchange rate. However, it would be more realistic to be able to solve a model allowing for the market to notice and react to large price-moving interventions by the Central Bank. By sovling this problem, Kercheval and Moreno [8] assumed that the rate dynamics changes to a different process for a random period after each intervention, after which it reverts to the pre-intervention process. They also gave an explicit numerical solution as well as the comparative statics analysis about the effects of the changes of parameters on the optimal intervention strategy.

Instead of only considering the intervention in purchasing and selling reserves in the foreign exchange market [4, 5, 6, 1, 8], Mundaca and Oksendal [9] and Cadenillas and Zapatero [2] set up a model taking account of both reserves and interest rates. Mundaca and Oksendal gave some general analysis under some general assumptions, while Cadenillas and Zapatero provide the exact numerical solution by using a geometric brownian motion to model the exchange rate and assuming the interest rate is another stochastic process.

The structure of this PCE paper is as follows. Section 2 gives a brief review of the classical stochastic control. In section 3, I formulate the impulse control problem and introduce the main theorem by illustrating Cadenillas and Zapatero's paper [1], but the proof of the theorem is modified here using the idea in Kercheval and Moreno [8]. I also demostrate the numerical algorithm and give several examples which have different parameters from those in Cadenillas and Zapatero [1]. And then in section 4, I give some related results in some other papers I mentioned above [5, 6]. Finally, I list some possible directions of future reading interest in section 5.

2 A Brief Review of the Instantaneous/Classical Stochastic Control Problem

In this section we give a brief formulation of the instantaneous stochastic control problem and state some main results for solving this problem (without proof).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space, on which is defined an $\mathcal{F}_t$-adapted m-dimensional standard Brownian motion $W(t)$. We consider the following stochastic controlled system:

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \quad t \in [0, T] \\ X(0) = x \in \mathbb{R}^n \end{cases} \tag{1}$$

where $T > 0$, the maps $b : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$, and U is a given Polish space (i.e. complete separable metric space).
Define a cost functional

$$J(u(\cdot)) = E\left[\int_0^T f(t, X(t), u(t))dt + h(X(T))\right] \tag{2}$$

where $f : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$.
Define the set of admissible controls

$$\mathcal{U}[0, T] = \{u : [0, T] \times \Omega \to U \mid \text{u is } \mathcal{F}_t\text{-adapted}\} \tag{3}$$

The function $u(\cdot)$ is called the control representing the action, decision, or policy of the decision-makers (controller). For example, $u(\cdot)$ could be the interest rate process in the exchange rate intervention model.

The instantaneous/classical optimal stochastic control problem can be stated as follows.
Problem (S) Minimize (2) subject to the state equation (1) over $\mathcal{U}[0, T]$.

For any $(s, y) \in [0, T]$, consider the state equation,

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dW(t), \quad t \in [s, T] \\ X(s) = y \end{cases} \tag{4}$$

along with the cost functional

$$J(s, y; u(\cdot)) = E\left[\int_s^T f(t, X(t), u(t))dt + h(X(T))\right] \tag{5}$$

We can define the following function,

$$\begin{cases} V(s, y) = \inf_{u(\cdot)\in\mathcal{U}[s,T]} J(s, y; u(\cdot)), \quad \forall(s, y) \in [0, T) \times \mathbb{R}^n \\ V(T, y) = h(y), \quad \forall y \in \mathbb{R}^n \end{cases} \tag{6}$$

Which is called the **value function** of the original Problem (S).

The following two theorems give an insight of how to solve for the value function. (see [10])

Theorem 2.1 (Bellman's principle of optimality)
For any $(s, y) \in [0, T) \times \mathbb{R}^n$,

$$V(s,y) = \inf_{u(\cdot)\in\mathcal{U}[s,T]} E\left[\int_s^{s'} f(t, X(t; s, y, u(\cdot)))dt + V(s', X(s'; s, y, u(\cdot)))\right], \tag{7}$$

$\forall\ 0 \leq s \leq s' \leq T$

Remark: This principle roughly means that if $u(\cdot)$ is optimal on $[s, T]$, then it is also optimal on $[s', T]$.

Equation (7) is called the **dynamic programming equation**. This equation is very difficult to handle, since the operation involved on the right-hand side of (7) is too complicated. Thus, we would like to explore (7) further, trying to get an equation for $V(s, y)$ with a simpler form. It turns out that this can be done to some extent. The following result gives a partial differential equation that a continuously differential value function should satisfy. We let $C^{1,2}([0, T] \times \mathbb{R}^n)$ be the set of all continuous functions $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that v_t, v_x and v_{xx} are all continuous in (t, x)

Theorem 2.2 Suppose the value function $V \in C^{1,2}([0, T] \times \mathbb{R}^n)$. Then V is a solution of the following terminal value problem of a second-order partial differential equation:

$$\begin{cases} -v_t + \sup_{u\in U} G(t, x, u, -v_x, -v_{xx}) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n \\ v|_{t=T} = h(x), & x \in \mathbb{R}^n \end{cases} \tag{8}$$

where

$$G(t, x, u, p, q) = \frac{1}{2}\mathrm{tr}(q\sigma(t, x, u)\sigma(t, x, u)^T) + \langle p, b(t, x, u)\rangle - f(t, x, u), \tag{9}$$

$$\forall\ (t, x, u, p, q) \in [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{S}^n$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a square matrix, $\langle\cdot, \cdot\rangle$ denotes a inner product and $\mathbb{S}^n$ is the set of all $n \times n$ (real) matrices.

We call (8) the **Hamilton-Jacobi-Bellman equation** (**HJB equation** for short) of Problem (S). The function $G(t, x, u, p, q)$ defined by (9) is called the **generalized Hamiltonian**.

Now let's move from the instantaneous control to the impulse control problem which consists of the following 3 sections.

3 Impulse Stochastic Control Problem - Optimal Central Bank Intervention in the Foreign Exchange Market

You can find most of this part from Cadenillas and Zapatero [1].

3.1 Problem Statement

Consider a probability space $(\Omega, \mathcal{F}, P)$ together with a filtration $\mathcal{F}_t$ generated by a one-dimensional Brownian motion W_t.
Denote

$$X(t) = \text{ domestic currency units per unit of foreign currency at time } t$$

Assume that in the absence of intervention, $X(t)$ follows the diffusion process

$$dX_x(t) = \mu(X_x(t))dt + \sigma(X_x(t))dW_t, \tag{10}$$
$$X_x(0) = x$$

where $\mu(\cdot) : \mathbb{R} \to \mathbb{R}$ and $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ are the functions satisfying the usual requirements for the exsistence and uniqueness of the solution to the diffussion process.
For example, $X(t)$ could be a geometric brownian motion

$$dX_x(t) = \mu X_x(t)dt + \sigma X_x(t)dW_t, \tag{11}$$
$$X_x(0) = x$$

where μ and σ are two constants.

Definition 3.1.1 An impulse control

$$\nu = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$$

is a sequence of intervention times τ_i and intervention sizes ξ_i, where τ_i is an infinite increasing sequence of stopping times with respect to the filtration $\mathcal{F}_t$

$$0 \leq \tau_1 < \tau_2 < \tau_3 < \tau_4 \cdots \tag{12}$$

and each $\xi_i : \Omega \to \mathbb{R}$ is $\mathcal{F}_{\tau_i}$ measurable.

Define the controlled process by $X_x^\nu(t)$ in two cases:
Case 1: if $\tau_1 > 0$

$$X_x^\nu(t) = X_x(t), 0 \leq t < \tau_1$$
$$X_x^\nu(\tau_i) = X_x^\nu(\tau_i^-) + \xi_i, i = 1, 2, 3, \ldots$$
$$X_x^\nu(t) = X_{X_x^\nu(\tau_i)}(t), \tau_i \leq t < \tau_{i+1}$$

Case 2: if $\tau_1 = 0$

$$X_x^\nu(\tau_1^-) = x, \text{ (convention)}$$

$$X_x^\nu(\tau_i) = X_x^\nu(\tau_i^-) + \xi_i, i = 1, 2, 3, \ldots$$
$$X_x^\nu(t) = X_{X_x^\nu(\tau_i)}(t), \tau_i \le t < \tau_{i+1}$$

This means that the effect of the control is to shift the process without affecting either the drift or the volatility. So after the intervention, the process follows the original diffusion dynamics until the controller decides to intervene again.

The Central Bank's goal is to select a control ν which minimizes the functional J defined by

$$J^\nu(x) = E\left[\int_0^\infty e^{-rt} f(X_x^\nu(t))dt + \sum_{i=1}^\infty e^{-r\tau_i} g(\xi_i) I_{\{\tau_i<\infty\}}\right] \tag{13}$$

where $f(\cdot) : \mathbb{R} \to \mathbb{R}$ is the running cost function, $g(\cdot) : \mathbb{R} \to \mathbb{R}$ is the intervention cost function, r is the discount rate, and $I(\cdot)$ is the indicator function. Note that the variable x is the starting value of the controlled process $X_x^\nu(t)$.

Here we could take $f(\cdot)$ and $g(\cdot)$ as follows,

$$f(x) = (x - \rho)^2$$
$$g(\xi) = \begin{cases} C + c\xi, & \xi > 0 \\ \min(C, D), & \xi = 0 \\ D - d\xi, & \xi < 0 \end{cases}$$
$$C, c, D, d, \rho \in (0, \infty) \tag{14}$$

where f represents the running cost incurred by deviating from the aimed exchange rate ρ. C and c represents the fixed and proportional cost per intervention when the Central Bank pushes the exchange rate upwards, respectively. D and d represents the fixed and proportional cost per intervention when the Central Bank pushes the exchange rate downwards, respectively.

Definition 3.1.2 An impulse control $\nu = (\tau_1, \tau_2, \ldots, \tau_n, \ldots; \xi_1, \xi_2, \ldots, \xi_n, \ldots)$ is called admissible if we have

$$X_x^\nu(\tau_i) > 0 \text{ for all } i \tag{15}$$
$$E\left[\int_0^\infty e^{-rt} f(X_x^\nu(t))dt\right] < \infty \tag{16}$$
$$P(\left[\lim_{i\to\infty} \tau_i \le t\right]) = 0, \forall t \ge 0 \tag{17}$$
$$\lim_{t\to\infty} E\left[e^{-rt} X_x^\nu(t)\right] = 0 \tag{18}$$

We will denote the set of admissible controls by Γ.

Definition 3.1.3 Define the value function $V(x)$

$$V(x) = \inf_{\nu\in\Gamma} J^\nu(x) \tag{19}$$

Where J is defined as in (13). Now our central problem becomes to solve the optimization problem (19), which means we want to find the value function $V(x)$ and the associated optimal control ν.

3.2 QVI and Verification Theorem

Definition 3.2.1 For a function $\phi : \mathbb{R}^+ \to \mathbb{R}$, and $x \in \mathbb{R}^+$, $\xi \in \mathbb{R}$, define the optimal intervention operator $\mathcal{M}$ as follows,

$$\mathcal{M}\phi(x) = \inf_{\xi} \left\{ g(\xi) + \phi(x+\xi) : \xi \in \mathbb{R}, \xi + x \in \mathbb{R}^+ \right\} \tag{20}$$

where $g(\cdot)$ is the intervention cost function defined as in (13).

We also need the partial differential operator $\mathcal{L}$ given by

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\phi(x) + \mu(x)\frac{\partial}{\partial x}\phi(x) - r\phi(x) \tag{21}$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are defined as in (10), and r is the discount rate defined as in (13). So specifically, if in the absence of intervention, the exchange rate follows the geometric brownian motion defined as in (11), then the operator $\mathcal{L}$ should be as follows,

$$\mathcal{L}\phi(x) = \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}\phi(x) + \mu x\frac{\partial}{\partial x}\phi(x) - r\phi(x) \tag{22}$$

Definition 3.2.2 We say that the function ϕ satisfies the quasi-variational inequalities (QVI) associated with problem (19) if ϕ satisfies the the following three conditions:

$$\mathcal{L}\phi(x) + f(x) \geq 0 \tag{23}$$

$$\phi(x) \leq \mathcal{M}\phi(x) \tag{24}$$

$$(\mathcal{L}\phi(x) + f(x))(\phi(x) - \mathcal{M}\phi(x)) = 0 \tag{25}$$

Note that the QVI plays a role similar to that of the HJB-equation in the instantanueous stochastic control.
We will show that the solution to the QVI above is exactly the solution to the optimal problem (19) if some conditions are satisfied. Before proving it, we can construct the following impulse control from the solution to the QVI:

Definition 3.2.3 Let ϕ be a continuous solution of the QVI defined above. Then the following impulse control is called the QVI-control associated to ϕ (if it exists):

$$\tau_1 = \inf \left\{ t \geq 0 : \phi(X_x^\nu(t^-))) = \mathcal{M}\phi(X_x^\nu(t^-))) \right\} \tag{26}$$

$$\tau_i = \inf \left\{ t > \tau_{i-1} : \phi(X_x^\nu(t^-))) = \mathcal{M}\phi(X_x^\nu(t^-))) \right\}, \ i = 2, 3, 4, \ldots \tag{27}$$

$$\xi_i = \arg \inf_{\xi} \left\{ g(\xi) + \phi(X_x^\nu(\tau_i^-) + \xi) : \xi \in \mathbb{R}, \xi + X_x^\nu(\tau_i^-) \in \mathbb{R}^+ \right\}, \ i \in \mathbb{N} \tag{28}$$

This means that the Central Bank intervenes whenever ϕ and $\mathcal{M}\phi$ conicide and the size of the intervention corresponds to $\mathcal{M}\phi$. Note that we make a covention that $X_x^\nu(0^-) = x$ in equation (26).

Theorem 3.2.4 Let $\phi \in C^1(\mathbb{R})$ be a solution of the QVI associated with the problem (19), and suppose there is a finite subset $\mathcal{N} \subset \mathbb{R}$ such that $\phi \in C^2(\mathbb{R}^+ - \mathcal{N})$. If ϕ satisfies the growth conditions

$$E\left[\int_0^\infty (e^{-rt}\sigma(X_x^\nu(t))\phi^{'}(X_x^\nu(t)))^2 dt\right] < \infty \tag{29}$$

$$\lim_{t\to\infty} E\left[e^{-rt}\phi(X_x^\nu(t))\right] = 0 \tag{30}$$

for every process $X_x^\nu(t)$ corresponding to an admissible impulse control ν, then for every $x \in \mathbb{R}^+$

$$V(x) \geq \phi(x).$$

Moreover, if the QVI-control corresponding to ϕ is admissible then it is an optimal impulse control, and for every $x \in \mathbb{R}^+$

$$V(x) = \phi(x)$$

where $V(x)$ is the value function defined in (19).

Proof: Consider any admissible control $\nu = \{(\tau_n, \xi_n)\}_{n\in\mathbb{N}}$. Define $\tau^*(t) = \max\{\tau_i : \tau_i \leq t\}$; note that almost surely, $\tau^*(t) \to \infty$ as $t \to \infty$, due to the admissibility condition (17). We can write

$$\begin{aligned} & e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t))) - \phi(x) \\ = & \sum_{i=1}^\infty I_{\{\tau_i \leq t\}}\left[e^{-r\tau_i}\phi(X_x^\nu(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^\nu(\tau_{i-1}))\right] \\ + & \sum_{i=1}^\infty I_{\{\tau_i\leq t\}} e^{-r\tau_i}\left[\phi(X_x^\nu(\tau_i)) - \phi(X_x^\nu(\tau_i^-))\right] \end{aligned}$$

Note that here, we make a convention that $X_x^\nu(0^-) = x$ and $e^{-r\tau_0}\phi(X_x^\nu(\tau_0)) = \phi(x)$. Between τ_{i-1} and τ_i^-, $X_x^\nu(t)$ actually follows the diffusion process (10), so an application of Ito's fomula (see Oksendal [9], theorem 4.1.2) gives

$$\begin{aligned} & e^{-r\tau_i}\phi(X_x^\nu(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^\nu(\tau_{i-1})) \\ = & \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\mathcal{L}\phi(X_x^\nu(s))ds + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s \end{aligned}$$

By the inequality (23), this expression becomes

$$\begin{aligned} & e^{-r\tau_i}\phi(X_x^\nu(\tau_i^-)) - e^{-r\tau_{i-1}}\phi(X_x^\nu(\tau_{i-1})) \\ \geq & \int_{\tau_{i-1}}^{\tau_i} e^{-rs}(-f(X_x^\nu(s)))ds + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s \end{aligned}$$

If τ_{i-1} and τ_i are the intervention times defined in (26) and (27), then $\phi(X_x^\nu(s))) < \mathcal{M}\phi(X_x^\nu(s))$ for $\tau_{i-1} \leq s < \tau_i$, so $\mathcal{L}\phi(X_x^\nu(s)) + f(X_x^\nu(s)) = 0$ by definition 2.2.2

of QVI. So the inequality above becomes an equality for the QVI-control associated to ϕ. Note that $\xi_i = X_x^\nu(\tau_i) - X_x^\nu(\tau_i^-)$, according to inequality (24), we have

$$e^{-r\tau_i}\left[\phi(X_x^\nu(\tau_i)) - \phi(X_x^\nu(\tau_i^-))\right] \geq -e^{-r\tau_i} g(\xi_i)$$

Also, this inequality becomes an equality for the QVI-control associated to ϕ, since $g(\xi_i) + \phi(X_x^\nu(\tau_i^-)) = \mathcal{M}\phi(X_x^\nu(\tau_i^-)) = \phi(X_x^\nu(\tau_i))$ if (τ_i, ξ_i) is the impulse control defined in definition 2.2.3 of QVI-control. Therefore, combining the above two inequalities, we obtain

$$\begin{aligned}
& \phi(x) - e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t))) \\
\leq & \sum_{i=1}^{\infty} I_{\{\tau_i \leq t\}} \left\{ e^{-r\tau_i} g(\xi_i) + \int_{\tau_{i-1}}^{\tau_i} e^{-rs}(f(X_x^\nu(s)))ds - \int_{\tau_{i-1}}^{\tau_i} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s \right\} \\
= & \sum_{i=1}^{\infty} e^{-r\tau_i} g(\xi_i) I_{\{\tau_i \leq t\}} + \int_0^{\tau^*(t)} e^{-rs}(f(X_x^\nu(s)))ds - \int_0^{\tau^*(t)} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s
\end{aligned}$$

Taking expectation, we have

$$\begin{aligned}
& \phi(x) - E\left[e^{-r\tau^*(t)}\phi(X_x^\nu(\tau^*(t)))\right] \\
\leq & E\left[\sum_{i=1}^{\infty} e^{-r\tau_i} g(\xi_i) I_{\{\tau_i \leq t\}} + \int_0^{\tau^*(t)} e^{-rs}(f(X_x^\nu(s)))ds - \int_0^{\tau^*(t)} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s\right]
\end{aligned}$$

Let t go to ∞, then $\tau^*(t) \to \infty$, so the left-hand side of the above inequality becomes $\phi(x)$ because of the growth condition (30). While the growth condition (29) implies that the expectation of the stochastic integral $\int_0^{\tau^*(t)} e^{-rs}\phi^{'}(X_x^\nu(s))\sigma(X_x^\nu(s))dW_s$ vanishes (see Oksendal [9], theorem 3.2.1). So we obtain

$$\phi(x) \leq E\left[\sum_{i=1}^{\infty} e^{-r\tau_i} g(\xi_i) I_{\{\tau_n < \infty\}} + \int_0^{\infty} e^{-rt} f(X_x^\nu(s))ds\right]$$

i.e.

$$\phi(x) \leq J^\nu(x)$$

As this is true for any control ν, we have

$$\phi(x) \leq V(x)$$

Again, it becomes an equality for the QVI-control associated to ϕ because all the above inequalites become equalities for the QVI-control associated to ϕ.

Remark: the proof of this verification theorem is a modification of the version in Cadenillas and Zapatero[1] using the idea in Moreno and Kercheval[8].

3.3 The Solution to the QVI

In this section we assume that in the absence of intervention, the exchange rate follows the geometric brownian motion defined in (11) and the running cost and intervention cost are defined as in (14). Next we are going to conjecture an impulse control and construct a function $V(x)$, and then verify that they solve the QVI inequalities, so by theorem 3.2.4, $V(x)$ is the value function and the impulse control conjectured is the optimal impulse control of problem (19).

Conjecture: The optimal impulse control is characterized by four parameters a, α, β, b,with $0 < a < \alpha \leq \beta < b < \infty$, such that it is optimal not to intervene while the process stays inside the band (a, b) and jump up to α when the process hits a, and jump down to β when the process hits b.
i.e.

$$\tau_1 = \inf\{t \geq 0 : X_x^\nu(t) \notin (a, b)\} \tag{31}$$

$$\tau_i = \inf\{t > \tau_{i-1} : X_x^\nu(t) \notin (a, b)\}\ , i = 2, 3, \ldots \tag{32}$$

$$X_x^\nu(\tau_i) = X_x^\nu(\tau_i^-) + \xi_i = \beta I_{\{X_x^\nu(\tau_i^-) \geq b\}} + \alpha I_{\{X_x^\nu(\tau_i^-) \leq a\}},\ \ i \in \mathbb{N} \tag{33}$$

If initially $x > b$, we should move the process immediately to β, similarly, if $x < a$ jump to α. Note here, we again make the convention that $X_x^\nu(0^-) = x$.

Let $V(x)$ denote the value function, then this strategy indicates that $V(x)$ should be of the form

$$V(x) = V(\alpha) + C + c(\alpha - x), \quad if\ \ x \in (0, a] \tag{34}$$

$$V(x) = V(\beta) + D + d(x - \beta), \quad if\ \ x \in [b, \infty) \tag{35}$$

If V were differentiable at a and b,then from equations (34)-(35),we would get

$$V^{'}(a) = -c \tag{36}$$

$$V^{'}(b) = d \tag{37}$$

By the definition of a and α in the conjecture above, we have $V(a) = \mathcal{M}V(a) = V(\alpha)+C+c(\alpha-a)$, which means the minimum of $V(y)+C+c(y-a)$ is attained at $y = \alpha$. So

$$\frac{d}{dy}[V(y) + C + c(y - a)] = 0$$

which implies

$$V^{'}(\alpha) = -c \tag{38}$$

Similaly, the minimum of $V(y) + D + d(b - y)$ is attained at $y = \beta$, we have

$$V^{'}(\beta) = d \tag{39}$$

We also conjecture that in the region (a,b), $V(x)$ satisfies

$$\mathcal{L}V(x) = -f(x) = -(x - \rho)^2, \quad if\ \ x \in (a, b) \tag{40}$$

Applying the standard methods of ordinary differential equations, we see the general solution (denoted by $h(x)$) to equation (40) is given by

$$h(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \frac{x^2}{-\sigma^2 - 2\mu + r} - \frac{2\rho x}{r - \mu} + \frac{\rho^2}{r} \tag{41}$$

where A and B are constants and

$$\gamma_1 = \frac{-\mu + \frac{1}{2}\sigma^2 - \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} \tag{42}$$

$$\gamma_2 = \frac{-\mu + \frac{1}{2}\sigma^2 + \sqrt{(\mu - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} \tag{43}$$

Note that the value function $V(x)$ has the properties (34)-(37) when $x \in [0, a) \cup [b, \infty)$, and $V(x)$ is exactly $h(x)$ when $x \in (a, b)$, by the continuity of $V(x)$ and $V^{'}(x)$ at the connecting points a and b, we have

$$h(a) = h(\alpha) + C + c(\alpha - a) \tag{44}$$

$$h(b) = h(\beta) + D + d(b - \beta) \tag{45}$$

$$h^{'}(a) = -c \tag{46}$$

$$h^{'}(b) = d \tag{47}$$

Again because $V(x)$ is exactly $h(x)$ when $x \in (a, b)$, so from the properties (38)-(39), we have

$$h^{'}(\alpha) = -c \tag{48}$$

$$h^{'}(\beta) = d \tag{49}$$

The six unknowns $A, B, a, b, \alpha, \beta$ can be obtained by solving the system of the above six equations (44)-(49).
The following theorem proves rigorously the above conjecture:

Theorem 3.3.1 Let $h(x)$ be defined as in (41)-(43), and let $A, B, a, b, \alpha, \beta$,with $0 < a < \alpha \leq \beta < b < \infty$ be a solution of the system of equations (44)-(49).Define the function $V : (0, \infty) \to [0, \infty)$ by

$$V(x) = \begin{cases} h(\alpha) + C + c(\alpha - x), & if \;\; x \leq a \\ h(x), & if \;\; a < x < b \\ h(\beta) + D + d(x - \beta), & if \;\; x \geq b \end{cases}$$

If

$$a < \frac{1}{2}\left[c(\mu - r) + 2\rho - \sqrt{(c(r - \mu) - 2\rho)^2 - 4(\rho^2 - rV(\alpha) - rC - rc\alpha)}\right] \tag{50}$$

$$b > \frac{1}{2}\left[d(\mu - r) + 2\rho - \sqrt{(d(r - \mu) - 2\rho)^2 - 4(\rho^2 - rV(\beta) - rD - rd\beta)}\right] \tag{51}$$

and

$$-c < V^{'}(x) < d, \quad \forall\ \alpha < x < \beta \tag{52}$$

$$V^{'}(x) \geq d, \quad \forall\ \beta \leq x \leq b \tag{53}$$

$$V^{'}(x) \leq -c, \quad \forall\ a \leq x \leq \alpha \tag{54}$$

then $V(x)$ is the Value Fuction of the problem (19), and the strategy (31)-(33) is the corresponding optimal impulse control.

Proof: First, we will show that V satisfies the QVI (23)-(25):
First Inequality:

$$\mathcal{L}V(x) + f(x) = \begin{cases} -c\mu x - r[h(\alpha) + C + c(\alpha - x)] + (x-\rho)^2 & if\ \ x \leq a \\ \mathcal{L}h(x) + (x-\rho)^2 & if\ \ a < x < b \\ d\mu x - r[h(\beta) + D + d(x-\beta)] + (x-\rho)^2 & if\ \ x \geq b \end{cases}$$

By construction of $h(x)$ we have that $\mathcal{L}V(x) + f(x) = 0$ in the interval (a, b). Condition (50) implies that $\mathcal{L}V(x)+f(x) > 0$ in $(0, a]$, and (51) implies $\mathcal{L}V(x)+f(x) > 0$ in $(0, a]$ in $[b, \infty)$.
Second Inequality:

$$\mathcal{M}V(x) = \begin{cases} h(\alpha) + C + c(\alpha - x) & if\ \ x \leq \alpha \\ h(x) + \min(C, D) & if\ \ \alpha < x < \beta \\ h(\beta) + D + d(x-\beta) & if\ \ x \geq \beta \end{cases}$$

We have used the condition (52) to obtain $\mathcal{M}V$ in the interval (α, β). Thus,$V(x) - \mathcal{M}V(x)$ is equal to zero in the intervention region $(0, a] \cup [b, \infty)$, and is negative in the region (a, b) because of the conditions (53)-(54).
Third Inequality: Follows automatically from above two.
Hence,V is a solution of the QVI.

Next, we show that V satisfies the growth conditions (29)-(30):
$V^{'}(x)$ is continuous in $[a, b]$ and is constant in $(0, a]$ and $[b, \infty)$, so $V(x)$ is bounded. Besides, note that σ is a constant in this example, so $V(x)$ satisfies the growth condition (29). Let ν be any admissible control, so it satisfies the admissibiltiy condition (18) $\lim_{t\to\infty} E\left[e^{-rt} X_x^{\nu}(t)\right] = 0$, so $V(x)$ satisfies the growth condition (30), since $V(x)$ is bounded in $[a, b]$ and just linear in $(0, a]$ and $[b, \infty)$.
So by theorem 3.2.4, $V(x)$ is the Value Function and the strategy (31)-(33) is the corresponding optimal impulse control.

3.4 Algorithm and Numerical Examples

This part describes Newton's Method to solve the nonlinear system of equations (44)-(49) for $A, B, a, b, \alpha, \beta$, and gives some explanations for the effects of the changes of different parameters on the optimal intervention strategy.

Define $F = (f_1, f_2, f_3, f_4, f_5, f_6) : \mathbb{R}^6 \to \mathbb{R}^6$, of which each component is as follows,

$$f_1(A, B, a, b, \alpha, \beta) = h(a) - h(\alpha) - C - c(\alpha - a)$$
$$f_2(A, B, a, b, \alpha, \beta) = h(b) - h(\beta) - D - d(b - \beta)$$
$$f_3(A, B, a, b, \alpha, \beta) = h^{'}(a) + c$$
$$f_4(A, B, a, b, \alpha, \beta) = h^{'}(b) - d$$
$$f_5(A, B, a, b, \alpha, \beta) = h^{'}(\alpha) + c$$
$$f_6(A, B, a, b, \alpha, \beta) = h^{'}(\beta) - d$$

The following algorithm gives Newton's method to solve for the "zero" of F.

Step 1: Input the initial guess for $(A, B, a, b, \alpha, \beta)$, denoted by a six-dimensional vector x.
Step 2: Evaluate F and the Jacobian matrix of F at x, denoted by a six-dimensional vector F(x) and 6×6 matrix $J_F(x)$, respectively.
Step 3: Use LU Factorization to solve the linear system $J_F(x)\delta x = -F(x)$ for the six-dimensional vector δx.
Step 4: $x \leftarrow x + \delta x$.
Step 5: Repeat step 2 to step 4 until the norm of δx is less than a pre-set tolorrence TOL.
Step 6: Output x.

Using the above algorithm, we obtained the results for $A, B, a, b, \alpha, \beta$ with different parameters in Tables 1-7. And by plugging back, we observe that the inequalities (50)-(54) are indeed satisfied, so theorem 3.3.1 applies.

Remark: During the experiment, we find that the initial guess in this algorithm is very important. If the initial guess is not appropriate, this algorithm would fail to find the answer. For example, in table 1, when the parameters are $\mu = 0.1, \sigma = 0.3, r = 0.06, \rho = 1.4, C = 0.5, c = 0.2, D = 0.7, d = 0.4$, if we set the initial guess as $A = 6, B = -50, a = 0.3, b = 3.0, \alpha = 1.4, \beta = 1.7$, the algorithm fails. However, if we set the initial guess as $A = 6, B = -80, a = 0.3, b = 3.0, \alpha = 1.4, \beta = 1.7$, the result showed in table 1 is obtained. One of the reasons why the algorithm fails for the first initial guess is Newton's method is an iterative method, and during the iteration, we get a negative number for a, but $h(x) = Ax^{\gamma_1} + Bx^{\gamma_2} + \frac{x^2}{-\sigma^2 - 2\mu + r} - \frac{2\rho x}{r - \mu} + \frac{\rho^2}{r}$ is undefined at a negative number, since γ_1 is also negative.

In Table 1 we analyze the effects of changes in the target rate ρ on the optimal strategy of the Central Bank. As expected, we observe that increase of the target rate can make the endogenous control band $[a, b]$ shift to the right and the optimal intervention point α and β increase.

We define the weak side of the target rate as the interval (ρ, ∞), which means in this range, the value of the domestic currency is "weak". And also define the strong side of the target rate as the interval $(0, \rho)$.

In Table 2 we analyze the effects of changes in μ on the optimal strategy of the Central Bank. When the drift of the exchange rate is 0, we observe that $\alpha < \rho < \beta$. However, when the domestic currency experiences devaluatory pressure ($\mu > 0$), we see that $\beta < \rho$. This means the Central Bank "overshoots" in the interventions on the weak side of the target rate. The opposite happens when the domestic currency tends to appreciate ($\mu < 0$). Besides, the higher the devaluatory presure, the sooner the Central Bank intervenes on the weak side of the target rate and the later on the strong side. Similarly, the higher the appreciatory presure, the sooner the Central Bank intervenes on the strong side of the target rate and the later on the weak side.

In Table 3 we study the effects of changes in volatility. We observe that the larger the volatility the higher the level of intervention on the weak side of the target rate, and the lower the level of the intervention on the strong side. This means that the Central Bank tends to wait for the exchange rate further apart from the target rate to intervene if the volatility gets larger.

In Table 4 and 5 we study the effects of changes in the fixed costs. As expected, when fixed costs increase, it is optimal to wait longer before intervening, although the sizes of the interventions will be larger.

In Table 6 and 7 we study the effects of the changes in proportional costs. When proportional costs increase, it is also optimal to wait longer.

4 Some Related Results

In this section, I report some more results related to the impulse control problem in [5, 6].

In [5], Korn gave an anologue to the Bellman's principle of the instantaneous stochastic control and constructed an iterative method to find the Value Function defined in section 3 as follows:
Let τ denote the first time to intervene after the start at x, denote

$$\mathcal{G}v(x) = \inf_{\tau \in \sum} E\left[e^{-r\tau} \mathcal{M}v(X_x(\tau^-)) + \int_0^\tau e{-}rtf(X_x(t))dt \right]$$

where $\sum$ is the set of finite stopping times and $X_x(t)$ is the uncontrolled process starting at x.

Theorem 4.1 Let f be a bounded, nonnegative continuous function and define

$$V_0(x) = E\left[\int_0^\infty e^{-rt} f(X_x(t))dt\right] \quad \forall\ x \in \mathbb{R}$$

Set

$$V_n(x) = \mathcal{G}V_{n-1}(x) \quad \forall n \in \mathbb{N}$$

Assume further that $\mathcal{M}V_0(x)$ is Lipschitz continuous and that the uncontrolled process $X_x(t)$ fulfills

$$\left|E[e^{-rt}X_x(t)] - E[e^{-rt}X_y(t)]\right| \leq C\,|x-y|$$

for a suitable constant $C > 0$. Then we have

$$V_n(x) \downarrow V(x) \quad \text{as } n \to \infty,$$

and

$$V(x) = \mathcal{G}V(x) \tag{55}$$

i.e., the value function V is a fixed point of the operator $\mathcal{G}$ and therefore the variant (55) of Bellman's principle holds.

Remark: Theorem 4.1 offers a computational algorithm to solve the impulse control problem: Solve a sequence of optimal stopping problem until this sequence converges. However, each step of this iteration involves a large amout of computations. So the practicality of this solution method relies crucially on the possibility to solve conventional optimal stopping problems in an efficient way.

In some models (such as the portfolio optimization model in [6]), one cannot guarantee sufficient regularity of the solution of the QVI to apply the verification theorem 3.2.4. To solve this problem, Korn [6] showed that the value function of the impulse control problem is a viscosity solution of the QVI.
The following reports his result. Note we concentrate on the QVI of Definition 3.2.2 when we will refer to "the QVI" below.

Definition 4.2 Let v be a continuous function. v is called a
(a) **viscosity subsolution** of the QVI if for all $\phi \in C^2$ with $\phi(\overline{x}) = v(x)$ and $v \leq \phi$ we have

$$\max\left\{-\left[\frac{1}{2}\sigma(\overline{x})^2\phi''(\overline{x}) + \mu(\overline{x})\phi'(\overline{x}) - rv(\overline{x}) + f(\overline{x})\right], v(\overline{x}) - \mathcal{M}v(\overline{x})\right\} \leq 0$$

(b) **viscosity supsolution** of the QVI if for all $\phi \in C^2$ with $\phi(\overline{x}) = v(x)$ and $v \geq \phi$ we have

$$\max\left\{-\left[\frac{1}{2}\sigma(\overline{x})^2\phi''(\overline{x}) + \mu(\overline{x})\phi'(\overline{x}) - rv(\overline{x}) + f(\overline{x})\right], v(\overline{x}) - \mathcal{M}v(\overline{x})\right\} \geq 0$$

(c) **viscosity solution** of the QVI if v is both a viscosity sub- and supersolution.

Theorem 4.3 (Viscosity property of the Value Function) Let the coefficients $\mu(\cdot)$, $\sigma(\cdot)$ of (10) be Lipschitz continuous and let the running cost function $f(\cdot)$ be polynomially bounded. Assume further that the value function V of (19) is continuous, polynomially bounded and satisfies the following Bellman's principle:

$$V(x) = \inf_{\nu \in \Gamma} E\left[e^{-r\tau} V(X_x^\nu(\tau)) + \int_0^\tau e^{-rt} f(X_x(t))dt + \sum_{i=1}^{\infty} e^{-r\tau_i} g(\xi_i) I_{\{\tau_i \leq \tau\}}\right]$$

for all finite stopping times τ. Then V is a viscosity solution of the QVI.

5 Future Directions of Interest : Impulse Control Under Stochastic Volatility

In the example of section 3, the volatility is deterministic. What if the volatility itself is a stochastic process instead of a constant? In Moreno's dissertation [7], the perturbation method employed by Fouque, Papanicolaou and Sircar [3] for solving the free-boundary problem of the American put option under stochastic volatility is modified for solving impulse control problems under stochastic volatilty. This would be my further reading topic.

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Table 1: Effects of Changes in ρ. Note: $\mu = 0.1$, $\sigma = 0.3$, $r = 0.06$, $C = 0.5$, $c = 0.2$, $D = 0.7$, $d = 0.4$

	ρ=1.2	ρ=1.4	ρ=1.5
a	0.4256	0.5513	0.6161
α	0.8997	1.0823	1.1739
β	1.0222	1.2264	1.3288
b	2.1155	2.3874	2.5215
A	-0.4670	-1.0251	-1.4423
B	-74.6369	-91.8556	-100.76497

Table 2: Effects of Changes in μ. Note: $\sigma = 0.3$, $r = 0.06$, $\rho = 1.4$, $C = 0.5$, $c = 0.2$, $D = 0.5$, $d = 0.2$

	μ=-0.2	μ=0.0	μ=0.1
a	0.6890	0.6441	0.5844
α	1.5618	1.3029	1.1553
β	1.7356	1.4311	1.2700
b	2.7361	2.4404	2.3152
A	-18.4276	-8.2536	-1.1669
B	-0.0080	60.2129	-92.6079

Table 3: Effects of Changes in σ. Note: $\mu = 0.0$, $r = 0.06$, $\rho = 1.4$, $C = 0.5$, $c = 0.2$, $D = 0.5$, $d = 0.2$

	σ=0.1	σ=0.3	σ=0.5
a	0.9217	0.6441	0.4762
α	1.3647	1.3029	1.2412
β	1.4138	1.4311	1.4415
b	1.9729	2.4404	2.7988
A	-3.1101	-8.2536	-11.6531
B	-1.0634	60.2129	38.6063

Table 4: Effects of Changes in C. Note: $\mu = 0.0, \sigma = 0.3$, $r = 0.06$, $\rho = 1.4$, $c = 0.2$, $D = 0.5$, $d = 0.2$

	C=0.3	C=0.5	C=0.7
a	0.6969	0.6441	0.6049
α	1.2594	1.3029	1.3345
β	1.3965	1.4311	1.4575
b	2.4171	2.4404	2.4589
A	-8.7146	-8.2536	-7.8673
B	60.1721	60.2129	60.2459

Table 5: Effects of Changes in D. Note: $\mu = 0.0, \sigma = 0.3$, $r = 0.06$, $\rho = 1.4$, $C = 0.5$, $c = 0.2$, $d = 0.2$

	D=0.3	D=0.5	D=0.7
a	0.6569	0.6441	0.6340
α	1.3380	1.3029	1.2762
β	1.4843	1.4311	1.3932
b	2.3367	2.4404	2.5212
A	-8.4357	-8.2536	-8.1064
B	60.1226	60.2129	60.2899

Table 6: Effects of Changes in c. Note: $\mu = 0.0$,$\sigma = 0.3$, $r = 0.06$, $\rho = 1.4$, $C = 0.5$, $D = 0.5$, $d = 0.2$

	c=0.0	c=0.2	c=0.4
a	0.6601	0.6441	0.6292
α	1.3414	1.3029	1.2688
β	1.4102	1.4311	1.4496
b	2.4262	2.4404	2.4532
A	-8.5377	-8.2536	-7.9871
B	60.1880	60.2129	60.2358

Table 7: Effects of Changes in d. Note: $\mu = 0.0$,$\sigma = 0.3$, $r = 0.06$, $\rho = 1.4$, $C = 0.5$, $c = 0.2$, $D = 0.5$

	d=0.0	d=0.2	d=0.4
a	0.6560	0.6441	0.6333
α	1.3355	1.3029	1.2745
β	1.4041	1.4311	1.4559
b	2.4183	2.4404	2.4613
A	-8.4234	-8.2536	-8.0968
B	60.1285	60.2129	60.2951