# Probability of a Row of Passengers Arriving in Sequence 

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## 1 Problem set up

I was boarding a plane yesterday, and as I approached my seat behind the other passengers I noticed my row was still empty. I was the last passenger to board, so I thought this was my lucky day! However, it turned out the guy in front of me, and the guy in front of him were exactly the other passengers assigned to my row. The guy furthest in front took the aisle and the one in front of me had the middle seat, while I of course had the window. As the first guy sat down and realized what was happening, he muttered "Murphy's Law." This got me thinking - how likely was it that all passengers assigned to a row arrive in consecutive order? Trying to work this out in flight turned out to be a great way to make time go by.

To be clear, the plane I was flying (Delta MD-88) uses $3 / 2$ seating, meaning there are 3 seats on one side of the aisle and 2 on the other, at least in coach. According to Seat Guru, there were 25 rows of such seating. I'm interested in the probability that, in any given row, the three passengers assigned to the side with 3 seats arrive consecutively, one after the other.

## 2 Simplification

### 2.1 One row with marbles

First let's consider just one row of $3 / 2$ seating (which would be a really awkward looking airplane). A nice abstraction to solve this situation is, given a bag with 5 marbles, 3 red and 2 black, what's the probability we draw the 3 red consecutively when drawing all 5 without replacement? First, there are

$$
\binom{5}{3,2}=\frac{5!}{3!2!}=10
$$

ways of drawing the 5 marbles. The notation above is the multinomial coefficient, and may be interpreted as, "the number of ways to draw 5 objects, 3 of which belong to category 1 and 2 of which belong to category 2." Here we think of "red" as category 1 and "black" as category 2. To draw the 3 red consecutively, there are only 3 combinations that could result:

$$
\begin{equation*}
r r r b b, b r r r b, b b r r r . \tag{1}
\end{equation*}
$$

Thus the probability is $\frac{3}{10}$.
Before moving on to two rows, let's figure out how to more generally count the combinations that give us consecutive draws, rather than attempting to list them as done above. To do this, we
treat the block of 3 red marbles as one object, say $R:=r r r$. Then instead of sampling from all combinations of
rrrbb,
which has 5 elements ( 3 in category $r$ and 2 in category $b$ ), we sample from all combinations of

$$
R b b,
$$

which has 3 elements ( 1 in category $R$ and 2 in category $b$ ). As an example, the combinations from line (1) look like

$$
R b b, b R b, b b R .
$$

So, the number of ways to draw them consecutively is the number of ways to draw 3 objects, 1 of which belongs to category $R$ and 2 of which belong to category $b$. In multinomial notation,

$$
\binom{3}{1,2}=\frac{3!}{1!2!}=3
$$

as before.

### 2.2 Two rows

We now add a second row of $3 / 2$ seating and ask, what's the probability that the three passengers assigned to the side with 3 seats arrive consecutively, in either the first or second rows? Again, consider marbles in a bag. Then the question becomes, given a bag with 10 marbles, of which we have 3 red, 2 black, 3 green, 2 white, what's the probability we draw the 3 red or 3 green consecutively?

First, there are

$$
\binom{10}{3,2,3,2}=\frac{10!}{3!2!3!2!}=25,200
$$

ways to draw the 10 marbles. Instead of attempting to list all possible combinations in which we get $r r r$ or $g g g$, we rely on the grouping idea presented in the 1-row case. First think of the 3 red as one object, letting $R:=r r r$ as before. Then, instead of sampling from all combinations of rrrbbgggww,
which has 10 elements $(3 r, 2 b, 3 g$, and $2 w)$, we are sampling from all combinations of
Rbbgggww,
which has 8 elements $(1 R, 2 b, 3 g$, and $2 w)$. Thus there are

$$
\binom{8}{1,2,3,2}=1,680
$$

ways to draw the 3 red marbles consecutively. Similarly, we label the 3 green marbles as $G:=g g g$, and sample from all combinations of
rrrbbGww.

As with the 3 red, there are

$$
\binom{8}{3,2,1,2}=1,680
$$

ways to draw the 3 green consecutively. Thus there are $2 \times 1,680=3,360$ ways to draw either $r r r$ or ggg, or both. We lastly need to subtract the instances in which we double counted occurrences, which are those in which we draw both rrr and $g g g$. How many ways are there to draw both in consecutive order? We now imagine sampling from all combinations of

$$
R b b G w w,
$$

and so there are

$$
\binom{6}{1,2,1,2}=180
$$

ways to draw both $r r r$ and $g g g$. Thus, the probability of drawing either 3 red or 3 green consecutively is

$$
\frac{3,360-180}{25,200} \approx 0.1262
$$

### 2.3 Generalizing to more rows

You can probably see we're using the inclusion-exclusion principle to count the number of ways to draw the marbles consecutively. In the 2-row case, we computed this as

$$
\begin{equation*}
\text { \# ways to draw } r r r+\text { \# ways to draw } g g g-\# \text { ways to draw } r r r \text { and } g g g \tag{2}
\end{equation*}
$$

and then divided this by the number of ways to draw the 10 marbles ( 25,200 ways). We had to subtract the last quantity because it was counted in both of the first quantities, so it was double counted. Let's give some notation for these events:

$$
\begin{aligned}
E_{R} & :=\{\text { draw } r r r\} \\
E_{G} & :=\{\text { draw } g g g\}
\end{aligned}
$$

Then, line (2) may be written in more familiar form as

$$
\left|E_{R}\right|+\left|E_{G}\right|-\left|E_{R} \cap E_{G}\right|,
$$

where $|\cdot|$ is set cardinality.
For 3 rows, for example, we'll have 15 marbles and 6 categories (dropping the color description as I've reach my limit of known colors), namely $3 c_{1}, 2 c_{2}, 3 c_{3}, 2 c_{4}, 3 c_{5}$, and $2 c_{6}$. The number of ways to draw either $3 c_{1}, 3 c_{2}, 3 c_{3}$, or any combination of these, is

$$
\begin{equation*}
\left|E_{1}\right|+\left|E_{2}\right|+\left|E_{3}\right|-\left|E_{1} \cap E_{2}\right|-\left|E_{1} \cap E_{3}\right|-\left|E_{2} \cap E_{3}\right|+\left|E_{1} \cap E_{2} \cap E_{3}\right| \tag{3}
\end{equation*}
$$

Again, the quantities being subtracted are each double-counted in the single event cases, so we subtract them out. But doing so also removes the intersection of all 3 events occurring, so we must
add it back in. This is the last quantity being added above. Line (3) is computed as

$$
\begin{align*}
& \underbrace{\binom{13}{1,3,3,2,2,2}}_{\left|E_{1}\right|}+\underbrace{\binom{13}{3,1,3,2,2,2}}_{\left|E_{2}\right|}+\underbrace{\binom{13}{3,3,1,2,2,2}}_{\left|E_{3}\right|} \\
& -\underbrace{\binom{11}{1,1,3,2,2,2}}_{\left|E_{1} \cap E_{2}\right|}-\underbrace{\binom{11}{1,3,1,2,2,2}}_{\left|E_{1} \cap E_{3}\right|}-\underbrace{\binom{11}{3,1,1,2,2,2}}_{\left|E_{2} \cap E_{3}\right|} \\
& +\underbrace{\binom{9}{1,1,1,2,2,2}}_{\left|E_{1} \cap E_{2} \cap E_{3}\right|}  \tag{4}\\
& =21,621,600+21,621,600+21,621,600 \\
& \quad-831,600-831,600-831,600 \\
& \quad+45,360 \\
& =62,415,360 .
\end{align*}
$$

Notice in Eqn. (4) that each of the quantities $\left|E_{i}\right|$ have the same number of ways of occurring, which is intuitive. Indeed, our sample space is

$$
c_{1} c_{1} c_{1} c_{2} c_{2} c_{3} c_{3} c_{3} c_{4} c_{4} c_{5} c_{5} c_{5} c_{6} c_{6}
$$

so the number of ways to draw $c_{1} c_{1} c_{1}$ is the same as the number of ways to draw $c_{2} c_{2} c_{2}$, as is $c_{3} c_{3} c_{3}$. Similarly, the quantities $\left|E_{i} \cap E_{j}\right|$ are the same for the same reason. Thus we may rewrite line (3) as

$$
3\left|E_{1}\right|-3\left|E_{1} \cap E_{2}\right|+\left|E_{1} \cap E_{2} \cap E_{3}\right|
$$

or more generally as

$$
\binom{3}{1}\left|E_{1}\right|-\binom{3}{2}\left|E_{1} \cap E_{2}\right|+\binom{3}{3}\left|E_{1} \cap E_{2} \cap E_{3}\right|
$$

Just for closure, we compute the probability in the 3-row case. There are

$$
\binom{15}{3,3,3,2,2,2}=756,756,000
$$

ways of choosing these marbles, so the 3 -row probability is

$$
\frac{62,415,360}{756,756,000} \approx 0.0825
$$

## 3 General Solution

We now solve the problem for $n$ rows. Following the set up for 3 rows, the inclusion exclusion principle we use for $n$ rows of $3 / 2$ seating is

$$
\begin{equation*}
\binom{n}{1}\left|E_{1}\right|-\binom{n}{2}\left|E_{1} \cap E_{2}\right|+\binom{n}{3}\left|E_{1} \cap E_{2} \cap E_{3}\right|-\ldots+(-1)^{n+1}\binom{n}{n}\left|\bigcap_{i=1}^{n} E_{i}\right| . \tag{5}
\end{equation*}
$$

In our marble abstraction, line (5) is the number of ways to draw 3 consecutive marbles of the same color out of $5 n$ marbles, of which 3 are color 1,2 are color 2,3 are color 3,2 are color $4, \ldots, 3$ are color $2 n-1$, and 2 are color $2 n$. In our context, this is the number of ways 3 passengers assigned to a row on the side of the plane with 3 seats arrive consecutively to their row, one after the other. We lastly need to compute the total number of ways the passengers may board the plane, which is

$$
\begin{equation*}
\binom{5 n}{3,2,3,2, \ldots, 3,2} \tag{6}
\end{equation*}
$$

where the 3,2 pattern is repeated $n$ times. Putting lines (5) and (6) together for 25 rows, we get that the probability is

$$
\frac{\binom{25}{1}\left|E_{1}\right|-\binom{25}{2}\left|E_{1} \cap E_{2}\right|+\binom{25}{3}\left|E_{1} \cap E_{2} \cap E_{3}\right|-\ldots+(-1)^{26}\binom{25}{25}\left|\bigcap_{i=1}^{25} E_{i}\right|}{\binom{125}{3,2,3,2, \ldots, 3,2}} \approx 0.0096
$$

or about a 1 in 104 chance. Thus, if you took 104 trips aboard a Delta MD-88, Murphy's Law is almost sure to happen.

