

Rectangle Rule

We must divide the interval from $x = a$ to $x = b$ into n equal subintervals of length $\Delta x = (b - a)/n$. Let c_k be any point in the k th subinterval.

Then

$$\int_a^b f(x) dx \approx M_n = f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$
$$\text{or } M_n = \Delta x [f(c_1) + f(c_2) + \cdots + f(c_n)]$$

For the rectangle (mid-point) rule, let c_k be the midpoint of the k th subinterval.

Here are four rectangle (mid-point) rule problems, completely worked out.

1. Use the rectangle (mid-point) rule, with $n = 4$, to approximate $\int_4^{12} \sqrt{x^2 - 1} dx$. Compute your answer to three decimal places.

$$M_4 = \Delta x (f(c_1) + f(c_2) + f(c_3) + f(c_4)) \Rightarrow a = 4; b = 12; n = 4; \Delta x = \frac{b - a}{n} = \frac{12 - 4}{4} = 2$$

$$f(x) = \sqrt{x^2 - 1}$$

$$[4, 6]$$

$$[6, 8]$$

$$[8, 10]$$

$$[10, 12]$$

$$c_1 = 5$$

$$c_2 = 7$$

$$c_3 = 9$$

$$c_4 = 11$$

$$f(c_1) = \sqrt{24} \quad f(c_2) = \sqrt{48} \quad f(c_3) = \sqrt{80} \quad f(c_4) = \sqrt{120}$$

$$M_4 = 2 (\sqrt{24} + \sqrt{48} + \sqrt{80} + \sqrt{120}) \approx 63.452$$

$$\int_4^{12} \sqrt{x^2 - 1} dx \approx 63.447$$

2. Use the rectangle (mid-point) rule, with $n = 2$, to approximate $\int_{-2}^4 (x - x^2) dx$.

$$M_2 = \Delta x (f(c_1) + f(c_2)) \Rightarrow a = -2; b = 4; n = 2; \Delta x = \frac{b-a}{n} = \frac{4 - (-2)}{2} = 3$$

$$f(x) = x - x^2$$

$$[-2, 1]$$

$$c_1 = -\frac{1}{2}$$

$$[1, 4]$$

$$c_2 = \frac{5}{2}$$

$$f(c_1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)^2$$

$$f(c_2) = \frac{5}{2} - \left(\frac{5}{2}\right)^2$$

$$= -\frac{1}{2} - \frac{1}{4} = -\frac{2}{4} - \frac{1}{4} = -\frac{3}{4}$$

$$= \frac{5}{2} - \frac{25}{4} = \frac{10}{4} - \frac{25}{4} = -\frac{15}{4}$$

$$M_2 = 3 \left(-\frac{3}{4} - \frac{15}{4} \right) = 3 \left(-\frac{18}{4} \right) = 3 \left(-\frac{9}{2} \right) = -\frac{27}{2}$$

$$M_2 = -13.5, \quad M_4 = -16.875, \quad M_8 = -17.71875, \quad M_{16} = -17.9296875$$

$$\int_{-2}^4 (x - x^2) dx = -18$$

3. Use the rectangle (mid-point) rule, with $n = 2$, to approximate $\int_{-4}^2 (x - 2x^2) dx$.

$$M_2 = \Delta x (f(c_1) + f(c_2)) \Rightarrow a = -4; b = 2; n = 2; \Delta x = \frac{b-a}{n} = \frac{2 - (-4)}{2} = 3$$

$$f(x) = x - 2x^2$$

$$[-4, -1]$$

$$c_1 = -\frac{5}{2}$$

$$[-1, 2]$$

$$c_2 = \frac{1}{2}$$

$$f(c_1) = -\frac{5}{2} - 2 \left(-\frac{5}{2} \right)^2$$

$$f(c_2) = \frac{1}{2} - 2 \left(\frac{1}{2} \right)^2$$

$$= -\frac{5}{2} - 2 \left(\frac{25}{4} \right) = -\frac{5}{2} - \frac{25}{2} = -\frac{30}{2} = -15$$

$$= \frac{1}{2} - 2 \left(\frac{1}{4} \right) = \frac{1}{2} - \frac{1}{2} = 0$$

$$M_2 = 3(-15 + 0) = 3(-15) = -45$$

$$M_2 = -45, \quad M_4 = -51.75, \quad M_8 = -53.4375, \quad M_{16} = -53.859375$$

$$\int_{-4}^2 (x - 2x^2) dx = -54$$

4. Use the rectangle (mid-point) rule, with $n = 3$, to approximate $\int_{-2}^1 (2x - 3x^2) dx$.

$$M_3 = \Delta x (f(c_1) + f(c_2) + f(c_3)) \Rightarrow a = -2; b = 1; n = 3; \Delta x = \frac{b-a}{n} = \frac{1-(-2)}{3} = 1$$

$$f(x) = 2x - 3x^2$$

$[-2, -1]$ $c_1 = -\frac{3}{2}$ $f(c_1) = 2\left(-\frac{3}{2}\right) - 3\left(-\frac{3}{2}\right)^2$ $= -3 - 3\left(\frac{9}{4}\right) = -3 - \frac{27}{4} = -\frac{39}{4}$	$[-1, 0]$ $c_2 = -\frac{1}{2}$ $f(c_2) = 2\left(-\frac{1}{2}\right) - 3\left(-\frac{1}{2}\right)^2$ $= -1 - 3\left(\frac{1}{4}\right) = -1 - \frac{3}{4} = -\frac{7}{4}$	$[0, 1]$ $c_3 = \frac{1}{2}$ $f(c_3) = 2\left(\frac{1}{2}\right) - 3\left(\frac{1}{2}\right)^2$ $= 1 - 3\left(\frac{1}{4}\right) = 1 - \frac{3}{4} = \frac{1}{4}$
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$$M_3 = 1\left(-\frac{39}{4} - \frac{7}{4} + \frac{1}{4}\right) = 1\left(-\frac{45}{4}\right) = -\frac{45}{4}$$

$$M_3 = -11.25, M_6 = -11.8125, M_{12} = -11.953125, M_{24} = -11.98828125$$

$$\int_{-2}^1 (2x - 3x^2) dx = -12$$