Notes on parametric surfaces and flux integrals

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A The normal vector to the graph of a function of two variables

A.1

Let $f(x,y)$ be a function of two variables. Its *graph* is a surface in 3-space, call it S. A point P on the graph can be identified by listing its coordinates as

$$P = (x, y, f(x,y))\,,$$

where the last one follows from the fact that the height of P must be given by the value of f if we want the point to be on S.

How to compute a vector perpendicular to S at P? The answer is to look at the gradient of a function $g(x,y,z)$ of *three* variables such that S is its zero-level surface. (The reason is that the gradient is always orthogonal to the level surfaces.) A simple choice for g is

$$g(x,y,z) = z - f(x,y)\,.$$

Computing the gradient, we have

$$\nabla g = -f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}\,.$$

We will call **normal** the choice of a *unit* orthogonal vector, so it is better to use:

$$\vec{n} = \frac{-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}}{\sqrt{1 + (f_x)^2 + (f_y)^2}}\,.$$

The above formula computes the normal vector to the graph S of $f(x,y)$ at any point P in the graph. Note that if we change (x,y) the point P will change, and the formula we have obtained is equally valid.

A.2

We want to use the last remark to change our point of view on how to interpret the previous calculation. The *position vector* corresponding to P is the vector

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + f(x,y)\vec{k}\,.$$

Observe that if we freeze either x or y, and we change the values of the other variable, the position vector $\vec{r}$ will trace for us a curve that is just the corresponding *cross section.* For example, if we set $x = a$, then for any y we get a point on the graph S corresponding to

$$a\vec{\imath} + y\vec{\jmath} + f(a,y)\vec{k}\,,$$

for all values of y. Thus as y changes, we get a *parametrized curve* where the parameter is precisely y. Same thing if we set $y = b$ and we let x change.

The upshot is that if we use alternatively x and y as parameters, we get two parametrized curves tracing all the cross sections. Each parametrized curve has its own velocity vector, which we get by taking the derivative of the position vector with respect to the appropriate parameter. If we consider x as a parameter (that is, variable) and keep y constant we get

$$\vec{r}_x = \frac{d}{dx}\vec{r} = \vec{\imath} + f_x\vec{k}\,,$$

while if we do the same with y variable and x constant we get

$$\vec{r}_y = \frac{d}{dy}\vec{r} = \vec{\jmath} + f_y\vec{k}\,.$$

It is important to note the following: both $\vec{r}_x$ and $\vec{r}_y$ are *tangent to S at P*, because they are velocity vectors, and the corresponding parametrized curves are entirely contained in the graph. It follows that if we take the cross-product we obtain a vector that is *orthogonal* to the graph, since by construction the cross-product is orthogonal to the plane determined by the vectors we are multiplying. In this case that plane is just the tangent plane to S.

Let us compute the cross product:

$$\vec{r}_x \times \vec{r}_y = (\vec{\imath} + f_x\vec{k}) \times (\vec{\jmath} + f_y\vec{k}) = -f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}\,. \tag{1}$$

We have re-obtained the previous result. Then if we want the normal vector (of unit magnitude) we can just write it as:

$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}\,.$$

B Parametric surfaces in general

The preceding discussion should have made clear that a surface can be described using *two parameters.* For the graph of a function $f(x, y)$ we used x, y. We can generalize that situation and say that we can describe a (parametrized) surface S in 3-space by considering the set of points P whose coordinates are functions of two parameters u, v:

$$P = (x(u, v), y(u, v), z(u, v))\,.$$

Equivalently, we can introduce the corresponding position vector as a function of two parameters:

$$\vec{r}(u, v) = x(u, v)\vec{\imath} + y(u, v)\vec{\jmath} + z(u, v)\vec{k}\,.$$

Thus the resulting surface S in 3-space is just the image of the (u, v)-plane (or a portion of it) through the three functions of two variables $x(u, v), y(u, v), z(u, v)$.

Freezing one of the two parameters, we get a *parametrized curve contained in the surface S.* If we set u equal to a constant, say $u = a$, then we obtain the parametrized curve

$$v \longmapsto \vec{r}(a, v) = x(a, v)\vec{\imath} + y(a, v)\vec{\jmath} + z(a, v)\vec{k}\,.$$

Similarly, setting $v = b$, we get another parametrized curve:

$$u \longmapsto \vec{r}(u, b) = x(u, b)\vec{\imath} + y(u, b)\vec{\jmath} + z(u, b)\vec{k}\,.$$

Each one of these parametrized curves has a velocity vector, which is tangent to the curve, and therefore it is tangent to the surface S as well. Therefore there are *two* velocity vectors $\vec{r}_u$, $\vec{r}_v$, that are computed by taking the derivatives of the components $x(u, v), y(u, v), z(u, v)$ of $\vec{r}(u, v)$ with respect to u and v, respectively. More explicitly:

$$\begin{aligned}
\vec{r}_u &= \frac{\partial x}{\partial u}\vec{\imath} + \frac{\partial y}{\partial u}\vec{\jmath} + \frac{\partial z}{\partial u}\vec{k} \\
\vec{r}_v &= \frac{\partial x}{\partial v}\vec{\imath} + \frac{\partial y}{\partial v}\vec{\jmath} + \frac{\partial z}{\partial v}\vec{k}
\end{aligned}$$

Once again, since $\vec{r}_u$ and $\vec{r}_v$ are tangent vectors to S at the point $\vec{r}(u,v)$, their cross product $\vec{r}_u \times \vec{r}_v$ will be orthogonal to S at the same point. It follows that if we want a normal *unit* vector at the point $\vec{r}(u,v)$, we may set

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \,.$$

Again, it should be noted that this vector depends on the point (u,v) and therefore it depends on the resulting point $\vec{r}(u,v)$ on the surface S.

C Examples

C.1 Cylinder of radius R

A cylinder of radius R and axis the z-axis is described by the equation

$$x^2 + y^2 = R^2 \,.$$

As it stands, this can *not* be the graph of a function $f(x,y)$.

First, let's say that the normal vector must be a vector of magnitude equal to one and orthogonal to the cylinder at any point. If $\vec{r} = x\vec{\imath} + y\vec{\jmath}$ is the position vector of a point in the cylinder, then we must have

$$\vec{n} = \frac{\vec{r}}{\|\vec{r}\|} \,.$$

But $\|\vec{r}\| = R$ on the cylinder, and using cylindrical coordinates, a point P on the cylinder will have coordinates

(2) $$x = R\cos\theta\,, \quad y = R\sin\theta\,, \quad z\,,$$

therefore we must have:

(3) $$\vec{n} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}.$$

Now let us use the general theory to confirm this result. We can use cylindrical coordinates (r,θ,z), and since on the cylinder the radius must be equal to R, we are left with the two parameters (θ,z). It follows that x,y,z in (2), *as functions of* θ *and* z, are a parametrization of the cylinder. By taking the derivatives with respect to θ and z we have:

$$\begin{aligned} \vec{r}_\theta &= -R\sin\theta\,\vec{\imath} + R\cos\theta\,\vec{\jmath} \\ \vec{r}_z &= \vec{k}\,. \end{aligned}$$

The cross product is:

$$\vec{r}_\theta \times \vec{r}_z = R\cos\theta\,\vec{\imath} + R\sin\theta\,\vec{\jmath}$$

so that $\|\vec{r}_\theta \times \vec{r}_z\| = R$, and we re-obtain (3).

C.2 Sphere of radius R

We have the equation:

$$x^2 + y^2 + z^2 = R^2 \,.$$

Once again, the position vector itself is orthogonal to the sphere, therefore we expect:

$$\vec{n} = \frac{\vec{r}}{\|\vec{r}\|} = \frac{1}{R}\vec{r}\,,$$

at any point P on the sphere with position vector $\vec{r}$.

C.2.1 Spherical coordinates

We can describe the sphere by using spherical coordinates with $\rho = R$:

$$x = R\,\sin\phi\,\cos\theta\,,\quad y = R\,\sin\phi\,\sin\theta\,,\quad z = R\,\cos\phi\,. \tag{4}$$

We can view the these relations as parametric equations for the sphere, where θ and ϕ are the parameters. (Recall that $0 \le \theta < 2\pi$ and $0 \le \phi \le \pi$!) Then by simply substituting the values in (4) into $\vec{n} = \vec{r}/R$ we should find:

$$\vec{n} = \sin\phi\,\cos\theta\,\vec{\imath} + \sin\phi\,\sin\theta\,\vec{\jmath} + \cos\phi\,\vec{k}\,. \tag{5}$$

To confirm this, let us use the parametric equations to compute the velocities and the normal:

$$\begin{aligned}\vec{r}_\phi &= R\cos\phi\,\cos\theta\,\vec{\imath} + R\cos\phi\,\sin\theta\,\vec{\jmath} - R\sin\phi\,\vec{k}\\ \vec{r}_\theta &= -R\sin\phi\,\sin\theta\,\vec{\imath} + R\sin\phi\,\cos\theta\,\vec{\jmath},\end{aligned}$$

and the cross product is:

$$\vec{r}_\phi \times \vec{r}_\theta = R^2\sin^2\phi\,\cos\theta\,\vec{\imath} + R^2\sin^2\phi\,\sin\theta\,\vec{\jmath} + R^2\sin\phi\,\cos\phi\,\vec{k}\,.$$

The magnitude is $\|\vec{r}_\phi \times \vec{r}_\theta\| = R^2\sin\phi$, so we obtain exactly (5).

C.2.2 Cartesian coordinates

To use the (x, y, z) coordinates we have to solve for one of the variables:

$$z = \sqrt{R^2 - x^2 - y^2}\,.$$

In this way we are limiting ourselves to the upper half. The other choice of the square root yields a function whose graph is the lower hemisphere. According to the initial formula,

$$\vec{n} = \frac{-f_x\vec{\imath} - f_y\vec{\jmath} + \vec{k}}{\sqrt{1 + (f_x)^2 + (f_y)^2}}\,.$$

We have:

$$\begin{aligned}f_x &= -\frac{x}{\sqrt{R^2 - x^2 - y^2}}\\ f_y &= -\frac{y}{\sqrt{R^2 - x^2 - y^2}}\\ 1 + (f_x)^2 + (f_y)^2 &= \frac{R^2}{R^2 - x^2 - y^2}\,,\end{aligned}$$

so that

$$\vec{n} = \frac{x}{R}\vec{\imath} + \frac{y}{R}\vec{\jmath} + \frac{\sqrt{R^2 - x^2 - y^2}}{R}\vec{k}\,.$$

This should be compared with (5)—using (4).

C.3 Upper half cone

Let S be the upper half cone with vertex angle $\pi/4$. An equation for it is

$$z = \sqrt{x^2 + y^2}\,.$$

Let us find parametric equations. We can use cartesian coordinates, since S is the graph of a function of two variables. The corresponding position vector would be:

$$\vec{r}(x, y) = x\vec{\imath} + y\vec{\jmath} + \sqrt{x^2 + y^2}\vec{k}\,. \tag{6}$$

Using cylindrical coordinates, the equation for the cone simply becomes $z = r$, so it follows that we can use r and θ as parameters:[1]

$$\vec{r}(r, \theta) = r\cos\theta\,\vec{\imath} + r\sin\theta\,\vec{\jmath} + r\,\vec{k}\,. \tag{7}$$

We can also use spherical coordinates (ρ, ϕ, θ). To obtain the cone we have to block ϕ to the value $\pi/4$, so the two parameters to use are ρ and θ. From the general expression for spherical coordinates we obtain

$$\vec{r}(\rho, \theta) = \frac{\sqrt{2}}{2}\rho\cos\theta\,\vec{\imath} + \frac{\sqrt{2}}{2}\rho\sin\theta\,\vec{\jmath} + \frac{\sqrt{2}}{2}\rho\,\vec{k}\,. \tag{8}$$

Note that (8) follows from (7) by replacing r with $\rho/\sqrt{2}$, as it should be.

From (7) we have

$$\begin{aligned} \vec{r}_r &= \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} + \vec{k} \\ \vec{r}_\theta &= -r\sin\theta\,\vec{\imath} + r\cos\theta\,\vec{\jmath} \end{aligned}$$

and

$$\vec{r}_r \times \vec{r}_\theta = -r\cos\theta\,\vec{\imath} - r\sin\theta\,\vec{\jmath} + r\,\vec{k}\,.$$

The magnitude is

$$\|\vec{r}_r \times \vec{r}_\theta\| = \sqrt{2}r\,,$$

so that the normal is:

$$\vec{n} = -\frac{1}{\sqrt{2}}\cos\theta\,\vec{\imath} - \frac{1}{\sqrt{2}}\sin\theta\,\vec{\jmath} + \frac{1}{\sqrt{2}}\,\vec{k}\,. \tag{9}$$

D Area vector

Given two vectors $\vec{v}$ and $\vec{w}$, their cross product has magnitude equal to the area of the parallelogram they determine. One could write A for the area, and therefore

$$\vec{v} \times \vec{w} = A\,\vec{n}\,,$$

where we have introduced a unit vector $\vec{n}$ directed in the same way as the cross product.

More generally, given a *flat* surface S with area A, we can choose an orientation, or in other words a normal $\vec{n}$. Then the **area vector** $\vec{A}$ is the vector with magnitude equal to the area and same direction as the chosen normal vector:

$$\vec{A} = A\,\vec{n}\,.$$

Now, if we have a parametrized surface S described by

$$\vec{r}(u, v) = x(u, v)\vec{\imath} + y(u, v)\vec{\jmath} + z(u, v)\vec{k}\,,$$

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[1] Note that $\vec{r}$ and r are distinct objects!

we have found that the normal vector can be written as

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}\,,$$

which we can rewrite as:

$$\vec{r}_u \times \vec{r}_v = \|\vec{r}_u \times \vec{r}_v\|\,\vec{n}\,.$$

This latter quantity should be thought of as an infinitesimal area vector carried by the surface. Indeed, for two very small increments Δu and Δv in the parameters (u, v), we can think of the vectors $\vec{r}_u\,\Delta u$ and $\vec{r}_v\,\Delta v$ as the sides of a very small parallelogram ΔA with vertex at $\vec{r}(u, v)$ and area vector:

$$\overrightarrow{\Delta A} = \vec{r}_u \times \vec{r}_v\,\Delta u \Delta v = \|\vec{r}_u \times \vec{r}_v\|\,\Delta u \Delta v\,\vec{n}\,.$$

E Flux through a surface and flux integrals

E.1 Flux through a flat surface

If S is a flat surface with normal vector $\vec{n}$ and area vector $\vec{A} = A\,\vec{n}$ and $\vec{F}$ is a *constant* vector field $\vec{F} = \vec{v}$, the **flux of $\vec{F}$ through** S is given by the number:

$$\vec{F} \cdot \vec{A} = \vec{v} \cdot \vec{n}\,A\,.$$

If we picture this constant vector field as a stream of particles (e.g. water or another fluid) all with constant velocity equal to $\vec{v}$, the flux above will tell us how many of them flow through S (per unit time). Note that only the component of $\vec{v}$ orthogonal to S matters: it is easy to realize that if S is placed in a way that its normal is orthogonal to the direction of $\vec{v}$, that is, $\vec{v}$ is tangent to S, there is no flux through S. So the dot product is essential, because it will precisely select the component of $\vec{v}$ orthogonal to S.

E.2 Definition of flux integral

What is the analogous concept if S is not flat? We also want to include the case where $\vec{F}$ is just any vector field, not just a constant one.

If we can break S in small pieces ΔS that are *approximately flat*, each with area ΔA, we can try to apply the above procedure piece by piece. So let us divide S in (approximately) flat pieces ΔS_i, each with area ΔA, and area vector $\Delta\vec{A}$. This means that we have chosen a normal vector for each piece. If we assume that $\vec{F}$ is approximately constant on each piece, the flux there will be

$$\text{Flux through piece} \approx \vec{F} \cdot \Delta\vec{A}\,,$$

and therefore

$$\text{Flux through } S \approx \sum \vec{F} \cdot \Delta\vec{A}\,,$$

so that actually

$$\text{Flux through } S = \lim_{\Delta A \to 0} \sum \vec{F} \cdot \Delta\vec{A}\,.$$

Therefore we can make the following definition:

The **flux integral** of the vector field $\vec{F}$ through the oriented surface S is

$$\int_S \vec{F} \cdot d\vec{A} = \lim_{\Delta A \to 0} \sum \vec{F} \cdot \Delta\vec{A}\,. \tag{10}$$

A good way to understand the flux integral is to think of $\vec{F}$ as the velocity vector field of some fluid. Then the flux integral is the rate of fluid through the surface S per unit time.

E.3 Flux integrals for parametrized surfaces

In order to *compute* flux integrals one has to make use of a concrete way to describe the surface. If we use a parametrization with position vector

$$\vec{r}(u,v) = x(u,v)\vec{\imath} + y(u,v)\vec{\jmath} + z(u,v)\vec{k}\,,$$

we have seen that the quantity:

$$\Delta\vec{A} = \vec{r}_u \times \vec{r}_v\,\Delta u \Delta v$$

can serve as the area vector of a small patch of the parametrized surface. If we use this in the definition (10) we have

$$\int_S \vec{F}\cdot d\vec{A} = \lim_{\Delta A\to 0}\sum \vec{F}\cdot\Delta\vec{A} = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \sum \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,\Delta u\Delta v\,.$$

The right hand side of this formula is a Riemann sum approximation of an ordinary double integral with respect to u and v, so we have:

(11) $$\int_S \vec{F}\cdot d\vec{A} = \int_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv\,,$$

where R is the region in the (u,v) plane corresponding to the surface S.

> EXERCISE: By using (1) and the examples in section C, check that formula (11) gives the formulas for the flux integrals on pages 886, 887 and 888 in the textbook

F Stokes' theorem

There is a certain relation between flux integrals and line integrals that generalizes Green's theorem.

First, let S be an oriented surface, and let the closed curve C be its boundary. We assume that the curve C is oriented in way compatible with the chosen orientation of S: traveling along C we should see S to our left. This is the same as applying the right hand rule: if the tip of the fingers of our right hand point in the same direction as the orientation of C, then the thumb should point in the same direction as the normal to S.

The statement is as follows:

> **Stokes' Theorem**
> Let S be a smooth oriented surface with boundary C. Let $\vec{F}$ be a smooth vector field defined on S and C. Then
> $$\int_C \vec{F}\cdot d\vec{r} = \int_S \operatorname{curl}(\vec{F})\cdot d\vec{A}$$
> The orientation of C is determined from the one of S by the right hand rule.

In other words, what the Stokes' Theorem says is that the circulation of $\vec{F}$ along C is equal to the flux of the *curl* of $\vec{F}$ through a surface whose boundary is C.