On cohomology classes related to uniformization, the Liouville action, and the dilogarithm

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Riemann Surfaces and constant $(=-1)$ negative curvature metrics

X compact Riemann surface, genus ≥ 2.

Conformal metric:

$$ds^2 = \rho\, dz \otimes d\bar{z}\,, \quad \rho \in C^\infty(U, \mathbb{R}_{>0})\,, \quad U \subset X \text{open.}$$

Line bundle with hermitian metric: (TX, ρ)

Liouville equation

Scalar curvature $K_\rho = -1$ iff

$$\boxed{\frac{\partial^2}{\partial z \partial \bar{z}} \log \rho = \frac{1}{2}\rho}$$

Relevant to uniformization

Find uniformizing metrics

Local extremum condition:

For a region $D \subset \mathbb{C}$ and $\rho \in C^\infty(D, \mathbb{R}_{>0})$ set:

$$S[\phi] = \frac{\sqrt{-1}}{2} \int_D \Big(\partial\phi \wedge \bar{\partial}\phi + e^\phi \, dz \wedge d\bar{z} \Big) , \qquad \phi = \log \rho$$

Then $(\sigma \in C^\infty(D, \mathbb{R})$ $)$:

$$K_\rho = -1 \iff dS[\phi](\sigma) \equiv \frac{d}{dt}\bigg|_{t=0} S[\phi + t\sigma] = 0$$

Globally on X:

- Definition of S is problematic
- Find an algebraic substitute for S
- Appropriate definition via secondary classes
- Computable?

The plan

- A crash course on Deligne cohomology
  - Include Hermitian metrics in the picture
- What happens on a compact Riemann surface
- Kleinian uniformizations
  - Classical dilogarithms
  - Relations with the Regulator class

Deligne cohomology

- Subring $A \subset \mathbb{R}$. Set $A(p) = (2\pi\sqrt{-1})^p A$.
- Deligne complex (X any complex manifold)

$$A(p)^\bullet_{\mathcal{D}} = A(p)_X \xrightarrow{\imath} \mathcal{O}_X \xrightarrow{d} \underline{\Omega}^1_X \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}^{p-1}_X$$

$$\xrightarrow{\simeq} \mathrm{Cone}\left(A(p)_X \oplus F^p\underline{\Omega}^\bullet_X \xrightarrow{\imath - \jmath} \underline{\Omega}^\bullet_X\right)[-1]$$

- Deligne Cohomology: $H^\bullet_{\mathcal{D}}(X, A(p)) = \mathbf{H}^\bullet(X, A(p)^\bullet_{\mathcal{D}})$
- Beĭlinson cup-product: a family

$$\cup_\alpha : A(p)^\bullet_{\mathcal{D}} \otimes A(q)^\bullet_{\mathcal{D}} \longrightarrow A(p+q)^\bullet_{\mathcal{D}}$$

induces a graded-commutative cup-product

$$H^i_{\mathcal{D}}(X, A(p)) \otimes H^j_{\mathcal{D}}(X, A(q)) \xrightarrow{\cup} H^{i+j}_{\mathcal{D}}(X, A(p+q))$$

Tame symbol map

Examples

- $\mathbb{Z}(1)^{\bullet}_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{O}_X^{\times}[-1] \Rightarrow H^{\bullet}_{\mathcal{D}}(X, \mathbb{Z}(1)) \cong H^{\bullet-1}(X, \mathcal{O}_X^{\times})$
  - $\mathcal{O}_X^{\times}(U) \cong H^1_{\mathcal{D}}(U, \mathbb{Z}(1))$ $\qquad f \sim (\log_i f, 2\pi\sqrt{-1}\; n_{ij})$
  - $\mathrm{Pic}(X) \cong H^2_{\mathcal{D}}(X, \mathbb{Z}(1))$ $\qquad L \sim (\log g_{ij}, 2\pi\sqrt{-1}\; c_{ijk})$
- $H^1_{\mathcal{D}}(X, \mathbb{R}(1))$ $\qquad f \sim (d\log f, \log|f|)$
- $\mathbb{Z}(2)^{\bullet}_{\mathcal{D}} \xrightarrow{\simeq} (\mathcal{O}_X^{\times} \to \underline{\Omega}^1_X)[-1]$
  - $H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \cong$ Isomorphism classes of invertible sheaves with connection
- Beĭlinson product & tame symbol

$$H^1_{\mathcal{D}}(U, \mathbb{Z}(1)) \otimes H^1_{\mathcal{D}}(U, \mathbb{Z}(1)) \ni f \otimes g \mapsto \big(f, g\big] \in H^2_{\mathcal{D}}(U, \mathbb{Z}(2))$$

$$\big(f, g\big] \sim (g^{n_{ij}}, \log_i f\, d\log g)$$

Steinberg relation: $\big(1 - f, f\big] \cong (\mathcal{O}_X^{\times}, d) \rightsquigarrow K_2(X)$

Hermitian metrics

- $L_\rho = (L, \rho)$ Invertible sheaf on X with hermitian fiber metric.

  Isomorphism classes: $\widehat{\mathrm{Pic}(X)}$

- Incorporating the hermitian structure: Hermitian-holomorphic Deligne cohomology

$$D(p)^\bullet_{h.h.} = \mathrm{Cone}\big(\mathbb{Z}(p)^\bullet_{\mathcal{D}} \oplus (F^p\underline{A}^\bullet_X \cap \sigma^{2p}\underline{\mathcal{E}}^\bullet_X(p)) \longrightarrow \mathbb{R}(p)^\bullet_{\mathcal{D}}\big)[-1]$$

$$H^\bullet_{\mathcal{D}_{h.h.}}(X,p) :\ = \mathbf{H}^\bullet(X, D(p)^\bullet_{h.h.})$$

- Thus: $\widehat{\mathrm{Pic}(X)} \cong H^2_{\mathcal{D}_{h.h.}}(X,1)$

- There is a Beĭlinson-type cup-product:

$$H^i_{\mathcal{D}_{h.h.}}(X,p) \otimes H^j_{\mathcal{D}_{h.h.}}(X,q) \xrightarrow{\cup} H^{i+j}_{\mathcal{D}_{h.h.}}(X,p+q)$$

  In particular for $L_\rho, {L'}_{\rho'} \in \widehat{\mathrm{Pic}(X)}$ we have

$$L_\rho \otimes {L'}_{\rho'} \longrightarrow \big(L_\rho, {L'}_{\rho'}\big] \in H^4_{\mathcal{D}_{h.h.}}(X,2)$$

On a curve

Back to a compact Riemann surface X (genus ≥ 2) with conformal metric ρ. We have $TX_\rho \in \widehat{\mathrm{Pic}(X)}$.

- Set: $\check{S}[\rho] := \big(TX_\rho, TX_\rho\big]$
- For dimensional reasons

$$\cdots \longrightarrow H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \longrightarrow H^4_{\mathcal{D}_{h.h.}}(X, 2) \longrightarrow 0$$

and furthermore $H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \cong H^2(X, \mathbb{R}(1))$ (since $\mathbb{R}(2)^\bullet_{\mathcal{D}} \xrightarrow{\simeq} \underline{\mathcal{E}}^\bullet_X(1)[-1]$.) Indeed:

$$\begin{tikzcd}
\mathbb{Z}(2) \arrow[r, "-\imath"] & \mathcal{O}_X \arrow[r, "-d"] \arrow[d, "\pi_1"] & \underline{\Omega}^1_X \arrow[d, "\pi_1"] & \\
 & \underline{\mathcal{E}}^0_X(1) \arrow[r, "-d"] & \underline{\mathcal{E}}^1_X(1) \arrow[r, "-d"] & \underline{\mathcal{E}}^2_X(1)
\end{tikzcd}$$

On a curve $\check{S}[\rho]$ is an $\mathbb{R}(1)$-valued class of degree 2

A geometric functional I

Let ω_ρ be the area form associated to the conformal metric ρ.

Theorem 1. *Let ρ be a conformal metric. Consider the class $S[\rho]\colon = \check{S}[\rho] - \frac{\sqrt{-1}}{2}[\omega_\rho]$. Then*

$$K_\rho = -1 \iff \left.\frac{d}{dt}\right|_{t=0} S[e^{t\sigma}\rho] = 0$$

- Choose a cover $\mathcal{U} \to X$
- A cocycle $\Omega_{\mathcal{U}}[\rho]$ representing the cup-product $\big(TX_\rho, TX_\rho\big]$ can be explicitly computed
- One verifies

$$\left.\frac{d}{dt}\right|_{t=0} \Omega_{\mathcal{U}}[e^{t\sigma}\rho] - \frac{\sqrt{-1}}{2}[\omega_{e^{t\sigma}\rho}] \equiv \sigma\big(c_1(\rho) - \sqrt{-1}\omega_\rho\big)\,,$$

where $\equiv$ means “up to total coboundary.”

A geometric functional II

Deligne's determinant of cohomology:

- To a pair of line bundles L, L' on X is assigned a $\mathbb{C}$-line generated by a symbol $\langle L\,,\,L'\rangle$. (Related to $\big(L, L'\big]$.)
- For two rational sections s and s' with disjoint divisors D and D', respectively, define ($\lVert\cdot\rVert$ denotes the metric for both bundles)

$$\log \lVert\langle s\,,\,s'\rangle\rVert^2 = \frac{1}{2\pi\sqrt{-1}}\int_X \partial\bar\partial \log\lVert s\rVert^2 \,\log\lVert s'\rVert^2 \\ + \log\lVert s\rVert^2[D'] + \log\lVert s'\rVert^2[D]$$

Theorem 2. *The cup product of (L,ρ) with (L',ρ') in hermitian holomorphic Deligne cohomology corresponds to the norm on the Deligne pairing $\langle L\,,\,L'\rangle$. The proportionality factor is $-\pi\sqrt{-1}$.*

However, no restrictions when the form $\big(L_\rho, L'_{\rho'}\big]$ is used!

Classical dilogarithm(s)

- Euler's dilogarithm

$$Li_2(z) = -\int^z \log(1-t)\,\frac{dt}{t}\,, \quad z \in \mathbb{P}^1 \setminus \{0,1,\infty\}$$

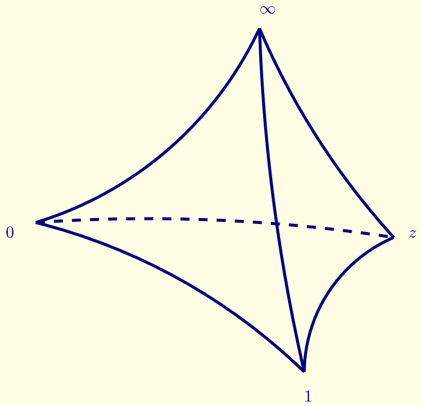
  - (Logarithm of a) Flat trivializing section of $\big(1-z, z\big]$

- Bloch-Wigner dilogarithm (Single valued!)

$$\mathscr{D}_2(z) = \arg(1-z)\,\log|z| + \operatorname{Im} Li_2(z)\,, \quad z \in \mathbb{P}^1 \setminus \{0,1,\infty\}$$

  - (Logarithm of) Hermitian square of $Li_2(z)$
  - Hyperbolic volume: $\mathscr{D}_2(z) = \mathrm{Vol}_{\mathbb{H}^3}(\Delta(\infty,0,1,z))$

$\Delta(\infty,0,1,z) =$



  - Regulator class: $H^3(\mathrm{PSL}_2(\mathbb{C})^\delta, \mathbb{R}(1)) \cong \mathbb{R}\,[\sqrt{-1}\mathscr{D}_2]$

Second kind Kleinian groups

$\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ geometrically finite second kind Kleinian group

- Discontinuity set $\Omega_\Gamma \subset \mathbb{P}^1$
- Riemann surface $X = \Omega_\Gamma/\Gamma \cong X_1 \sqcup \cdots \sqcup X_n$
- Compute cohomology with respect to the "cover" $\Omega_\Gamma \to X$
- Various associated 3-manifolds:

$$M = \mathbb{H}^3/\Gamma\,, \quad N = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma\,, \quad X = \partial N$$

Theorem 3. *Let* $\imath : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ *be the inclusion. The class* $\imath^*[\sqrt{-1}\mathscr{D}_2]$ *is the obstruction to closing the descent equations (i.e. calculating a cocycle) for* $\big(TX_\rho, TX_\rho\big]$ *with respect to the cover* $\Omega_\Gamma \to X$ *. This obstruction is zero for* Γ *of the second kind.*

Can be interpreted as: $\mathit{Transgr}([\sqrt{-1}\mathscr{D}_2]) = \big(TX_\rho, TX_\rho\big]$

Outlook

- Singular metrics
- Regularized volumes and relative classes
- Computations

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