

Abelian (2)-Gerbes, tame symbols, and hermitian structures

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- On hermitian-holomorphic classes related to uniformization, the dilogarithm and the Liouville Action <http://arXiv.org/math.CV/0211055>
- Hermitian-holomorphic gerbes and Deligne cohomology, and a class of tame symbols In preparation

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1 Tame symbols

Let $A = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Let X be a complex manifold (or an algebraic variety $/\mathbb{C}$). The **tame symbol** map comes from a **cup-product** operation in Deligne cohomology:

$$H_{\mathcal{D}}^i(X, A(p)) \otimes H_{\mathcal{D}}^j(X, A(q)) \xrightarrow{\cup} H_{\mathcal{D}}^{i+j}(X, A(p+q))$$

where $H_{\mathcal{D}}^{\bullet}(X, A(p)) = \mathbf{H}^{\bullet}(X, A(p)_{\mathcal{D}}^{\bullet})$ and the complex:

$$\begin{aligned} A(p)_{\mathcal{D}}^{\bullet} &= A(p)_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{d} \underline{\Omega}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}_X^{p-1} \\ &\xrightarrow{\cong} \text{Cone} \left(A(p)_X \oplus F^p \underline{\Omega}_X^{\bullet} \xrightarrow{i-j} \underline{\Omega}_X^{\bullet} \right) [-1] \end{aligned}$$

is the p^{th} **A -Deligne complex**. (Here $A(p) \stackrel{\text{def}}{=} (2\pi\sqrt{-1})^p A$.)

In particular: $A = \mathbb{Z}$, $p = q = 1$, then $\mathbb{Z}(1)_{\mathcal{D}}^{\bullet} \xrightarrow{\cong} \mathcal{O}_X^{\times}[-1]$ hence

$$\mathcal{O}_X^{\times}[-1] \otimes \mathcal{O}_X^{\times}[-1] \xrightarrow{\cup_{\alpha}} (\mathcal{O}_X^{\times} \xrightarrow{-d \log} \underline{\Omega}_X^1)[-1]$$

\Downarrow

$$\mathbf{H}^{i-1}(X, \mathcal{O}_X^{\times}) \otimes \mathbf{H}^{j-1}(X, \mathcal{O}_X^{\times}) \xrightarrow{\cup} \mathbf{H}^{i+j-1}(X, \mathcal{O}_X^{\times} \xrightarrow{d \log} \underline{\Omega}_X^1)$$

Tame symbol map for the corresponding geometric objects:

$$x \otimes y \longmapsto (x, y]$$

- $i = j = 1$ f, g invertible functions $\implies (f, g]$ \mathcal{O}_X^{\times} -torsor
- $i = 1, j = 2$ f invertible function, $L \in \text{Pic } X \implies (f, L]$ \mathcal{O}_X^{\times} -gerbe
- $i = j = 2$ $L, L' \in \text{Pic } X \implies (L, L']$ \mathcal{O}_X^{\times} -2-gerbe

plus a notion of **connection**

2 Geometric objects

2.1 Deligne's torsor

To define the tame symbol $(f, g]$ of $f, g \in H^0(X, \mathcal{O}_X^\times)$: for some cover $\mathcal{U}_X \rightarrow X$ extract $\log f$. Then:

- **Generator** $\{\log f, g\}$
 - **Relation** For $n \in \mathbb{Z}$ one sets: $\{\log f + 2\pi\sqrt{-1}n, g\} \equiv g^{-n}\{\log f, g\}$
 - **Connection** $\nabla\{\log f, g\} = -\frac{1}{2\pi\sqrt{-1}}\{\log f, g\} \otimes \log f \frac{dg}{g}$
1. $(g, f] \cong (f, g]^{\otimes -1}$
 2. $(f, g_1g_2] \cong (f, g_1] \otimes (f, g_2]$
 3. **Steinberg relation** in $K_2(X)$: let f and $1 - f$ be both invertible:

$$(1 - f, f] \cong (\mathcal{O}_X^\times, d)$$

2.2 Brylinski-McLaughlin's higher symbols

The \mathcal{O}_X^\times -gerbe $(f, L]$ (After Stackification (sic!))

- **Objects** On $U \rightarrow X$ an object $(f, s]$ is a (non-vanishing) section $s \in L|_U$.
- **Morphisms** If $s, s' \in L|_U \Rightarrow s' = s g$ with $g \in \mathcal{O}_X^\times(U)$.

$$\boxed{\underline{Hom}_U((f, s], (f, s']) \cong (f, g]|_U} \quad (f, s] \xrightarrow{(f, g]} (f, s'] \xrightarrow{(f, h]} (f, s'']$$

$$\underbrace{\hspace{10em}}_{(f, gh] \cong (f, g] \otimes (f, h]}$$

- **Connective structure** $(f, s] \rightsquigarrow$ trivial $\underline{\Omega}_U^1$ – torsor

$$\varphi \in \underline{Hom}_U((f, s], (f, s']) \rightsquigarrow [\omega \mapsto \omega + \varphi^{-1} \otimes \nabla \varphi]$$

$$g \in \underline{Aut}_U((f, s]) \simeq g \in \mathcal{O}_X^\times(U) \rightsquigarrow [\omega \mapsto \omega - \frac{dg}{g}]$$

The \mathcal{O}_X^\times -2-gerbe $(L, L']$ (After 2-Stackification $\ddot{\smile}$)

- **Objects** On $U \rightarrow X$ an object $(s, L']$ is a (non-vanishing) section $s \in L|_U$.
- **Morphisms** If $s, s' \in L|_U \Rightarrow s' = s g$ with $g \in \mathcal{O}_X^\times(U)$.
(Also, if $t, t' \in L'|_U \Rightarrow t' = t k, k \in \mathcal{O}_X^\times(U)$.)

$$\boxed{\begin{array}{c} \underline{\text{Hom}}_U((s, L'], (s', L']) \cong (g, L'] \\ \mathcal{O}_U^\times\text{-gerbe} \end{array}} \quad (s, L'] \begin{array}{c} \xrightarrow{g,t} \\ \Downarrow g,k \\ \xrightarrow{g,t'} \end{array} (s', L'] \begin{array}{c} \xrightarrow{h,t} \\ \Downarrow h,k \\ \xrightarrow{h,t'} \end{array} (s'', L']$$

- **Connectivity** $(s, L'] \rightsquigarrow$ (neutral) $\underline{\Omega}_U^1$ – gerbe

Connective structure on $(s, L']$:

- $(g, t] \rightsquigarrow \underline{\Omega}_U^1$ -torsor
- $(g, k]$: morphism of $\underline{\Omega}_U^1$ -torsors

3 Hermitian structures

3.1 Metrized line bundles

- $L_\rho = (L, \rho)$ Invertible sheaf on X with hermitian fiber metric.
Isomorphism classes: $\widehat{\text{Pic}} X$
- Deligne's observation: $\widehat{\text{Pic}} X \cong \mathbf{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\log|\cdot|} \underline{\mathcal{E}}_X^0)$
- In general: Hermitian-holomorphic Deligne cohomology

$$D(p)_{h.h.}^\bullet = \text{Cone}(\mathbb{Z}(p)_{\mathcal{D}}^\bullet \oplus (F^p \underline{A}_X^\bullet \cap \sigma^{2p} \underline{\mathcal{E}}_X^\bullet(p)) \longrightarrow \mathbb{R}(p)_{\mathcal{D}}^\bullet)[-1]$$

$$H_{\mathcal{D}_{h.h.}}^\bullet(X, p) := \mathbf{H}^\bullet(X, D(p)_{h.h.}^\bullet)$$

- Quasi-isomorphism: $D(1)_{h.h.}^\bullet \xrightarrow{\cong} (\mathcal{O}_X^\times \xrightarrow{-\log|\cdot|} \underline{\mathcal{E}}_X^0)[-1]$

$$\text{Thus: } \widehat{\text{Pic}} X \cong H_{\mathcal{D}_{h.h.}}^2(X, 1)$$

3.2 Higher order hermitian structures

Is there a geometric interpretation for the classes in $H_{\mathcal{D}_{h.h.}}^3(X, 1)$? (And also higher groups?) In other words, **gerbes with hermitian structure**

Definition 1. Hermitian structure on an \mathcal{O}_X^\times -gerbe $\underline{\mathbf{G}}$

Let $\underline{\mathcal{E}}_X^+$ be the sheaf of (smooth) positive real functions on X .

1. For each object P in $\underline{\mathbf{G}}_U$ assign a $\underline{\mathcal{E}}_X^+$ -torsor $\underline{h}(P)$
2. For each $f: P \rightarrow Q$ assign a **morphism of torsors** $f_*: \underline{h}(P) \rightarrow \underline{h}(Q)$
If $\varphi \in \underline{Aut}_U(P)$ then

$$\varphi_*: \underline{h}(P) \xrightarrow{\sim} \underline{h}(P), \quad h \longmapsto h \cdot |\varphi|^2$$

Theorem 1. *Equivalence classes of \mathcal{O}_X^\times -gerbes with hermitian structure are classified by*

$$\mathbf{H}^2(X, \mathcal{O}_X^\times \rightarrow \underline{\mathcal{E}}_X^0).$$

Definition 2. Type (1, 0) compatible connective structure

Let $F^1 \underline{A}^1$ the sheaf of smooth complex valued type (1, 0) forms.

A type (1, 0) connective structure on \underline{G} is the assignment to each object P of \underline{G}_U of a $F^1 \underline{A}_U^1$ -torsor $\underline{Co}(P)$

It is compatible with the hermitian structure if for each object P of \underline{G}_U there is a map of torsors $\underline{h}(P) \longrightarrow \underline{Co}(P)$, $r \longmapsto \nabla_r$ such that

$$r \cdot \rho \longmapsto \nabla_r + \partial \log \rho, \quad \rho \in \underline{\mathcal{E}}_X^+(U)$$

Theorem 2. Type (1, 0) connective structures on a \mathcal{O}_X^\times -gerbe \underline{G} compatible with a given hermitian structure are classified by the group $\mathbf{H}^3(X, D(1)_{h.h.}^\bullet)$.

The quasi-isomorphism: $D(1)_{h.h.}^\bullet \xrightarrow{\simeq} (\mathcal{O}_X^\times \xrightarrow{\log|\cdot|} \underline{\mathcal{E}}_X^0)[-1]$ implies:

Corollary 3. A compatible type (1, 0) connective structure structure on a gerbe \underline{G} is uniquely determined up to equivalence.

3.3 More hermitian constructions

- Similar interpretation for $H_{\mathcal{D}_{h.h.}}^4(X, 1) \cong \mathbf{H}^3(X, \mathcal{O}_X^\times \xrightarrow{\log|\cdot|} \underline{\mathcal{E}}_X^0)$
- Use a **modified cup-product**

$$\mathcal{O}_X^\times[-1] \otimes \mathcal{O}_X^\times[-1] \xrightarrow{\cup} (\mathcal{O}_X^\times \xrightarrow{-\log|\cdot|} \underline{\mathcal{E}}_X^0)[-1]$$

\Downarrow

$$H^{i-1}(X, \mathcal{O}_X^\times) \otimes H^{j-1}(X, \mathcal{O}_X^\times) \xrightarrow{\cup} \mathbf{H}^{i+j-1}(X, \mathcal{O}_X^\times \xrightarrow{\log|\cdot|} \underline{\mathcal{E}}_X^0)$$

Theorem 4. *The torsor $(f, g]$, the gerbe $(f, L]$ and the 2-gerbe $(L, L']$ are equipped with hermitian structures.*

e.g. for $f, g \in H^0(X, \mathcal{O}_X^\times)$ the Čech cocycle

$$\boxed{\left(g^{-n_{ij}}, -\frac{1}{2\pi} \arg_i(f) \log |g| \right)}$$

represents the metrized bundle in $\widehat{\text{Pic}} X$ corresponding to $(f, g]$.

- Cup product for Hermitian holomorphic Deligne cohomology

Since $D(p)_{h.h.}^\bullet = \text{Cone}(\dots)$ there is a (Beilinson) cup-product:

$$D(p)_{h.h.}^\bullet \otimes D(q)_{h.h.}^\bullet \xrightarrow{\cup_\alpha} D(p+q)_{h.h.}^\bullet$$

$$\Downarrow$$

$$H_{\mathcal{D}_{h.h.}}^i(X, p) \otimes H_{\mathcal{D}_{h.h.}}^j(X, q) \xrightarrow{\cup} H_{\mathcal{D}_{h.h.}}^{i+j}(X, p+q)$$

In particular for $L_\rho, L'_{\rho'} \in \widehat{\text{Pic}} X$ we have

$$\begin{array}{ccc} L_\rho \otimes L'_{\rho'} & \longrightarrow & (L_\rho, L'_{\rho'}) \in H_{\mathcal{D}_{h.h.}}^4(X, 2) \\ \downarrow & & \downarrow \text{forget} \\ L \otimes L' & \longrightarrow & (L, L') \in H_{\mathcal{D}}^4(X, \mathbb{Z}(2)) \end{array}$$

Type (2, 2) curving

4 Riemann surfaces and algebraic curves

X : compact Riemann surface ($g \geq 2$)

- For dimensional reasons $\cdots \longrightarrow H_{\mathcal{D}}^3(X, \mathbb{R}(2)) \longrightarrow H_{\mathcal{D}_{h.h.}}^4(X, 2) \longrightarrow 0$ and $H_{\mathcal{D}}^3(X, \mathbb{R}(2)) \cong H^2(X, \mathbb{R}(1))$ (since $\mathbb{R}(2)_{\mathcal{D}}^{\bullet} \xrightarrow{\cong} \underline{\mathcal{E}}_X^{\bullet}(1)[-1]$.) Indeed:

$$\begin{array}{ccccc}
 \mathbb{Z}(2) & \xrightarrow{-\iota} & \mathcal{O}_X & \xrightarrow{-d} & \underline{\Omega}_X^1 \\
 D(2)_{h.h.}^{\bullet} = & & \downarrow \pi_1 & & \downarrow \pi_1 \\
 & & \underline{\mathcal{E}}_X^0 \otimes_{\mathbb{R}} \mathbb{R}(1) & \xrightarrow{-d} & \underline{\mathcal{E}}_X^1 \otimes_{\mathbb{R}} \mathbb{R}(1) & \xrightarrow{-d} & \underline{\mathcal{E}}_X^2 \otimes_{\mathbb{R}} \mathbb{R}(1)
 \end{array}$$

- Let $L_{\rho}, L'_{\rho'} \in \widehat{\text{Pic}} X$:

For a curve X $(L_{\rho}, L'_{\rho'})$ defines an $\mathbb{R}(1)$ valued class of degree 2

- More generally, for a family $\pi: X \rightarrow S$ there is a trace map

$$H_{\mathcal{D}_{h.h.}}^4(X, 2) \longrightarrow H_{\mathcal{D}_{h.h.}}^2(S, 1)$$

4.1 Deligne's determinant of cohomology

- To a pair $L, L' \in \text{Pic } X$ is assigned a \mathbb{C} -vector space $\langle L, L' \rangle$.
 - s and s' : rational sections with disjoint divisors D and D' , respectively. $\langle s, s' \rangle$ is a generator $/\mathbb{C}$
 - f and g rational functions. **Relations:**

$$\langle fs, s' \rangle \equiv f(\text{div}(s')) \langle s, s' \rangle \quad \langle s, gs' \rangle \equiv g(\text{div}(s)) \langle s, s' \rangle$$

- If $L_\rho, L'_{\rho'} \in \widehat{\text{Pic } X}$ (denote the metric for both bundles by $\|\cdot\|$)

$$\log \|\langle s, s' \rangle\|^2 = \frac{1}{2\pi\sqrt{-1}} \int_X \partial\bar{\partial} \log \|s\|^2 \log \|s'\|^2 + \log \|s\|^2 [D'] + \log \|s'\|^2 [D]$$

defines a **Hermitian metric** on the \mathbb{C} vector space $\langle L, L' \rangle$: the **Determinant of cohomology**

Theorem 5. *The cup product of $(L_\rho, L'_{\rho'}) \in H^4_{\mathcal{D}_{h.h.}}(X, 2)$ in hermitian holomorphic Deligne cohomology corresponds to the norm $\|\cdot\|$ on the Deligne pairing $\langle L, L' \rangle$. The proportionality factor is $-\pi \sqrt{-1}$.*

4.2 Conformal metrics

Let ρ be a **conformal metric** on X . Same as:

$$T_{X\rho} \equiv (T_X, \rho) \in \widehat{\text{Pic}} X.$$

Set: $\check{S}_X[\rho] := -(2\pi\sqrt{-1})^{-1} \otimes (T_{X\rho}, T_{X\rho}) \in (2\pi\sqrt{-1})^{-1} \otimes H^4_{\mathcal{D}_{h.h.}}(X, 2)$

Theorem 6. *Set $S_X[\rho] := \check{S}_X[\rho] + \frac{1}{2\pi} \int_X \text{vol}_\rho$. We have:*

$ \begin{array}{l} \text{Scalar curvature} \\ K_\rho = -1 \end{array} \iff \frac{d}{dt} \Big _{t=0} S_X[e^{t\sigma} \rho] = 0 \quad \forall \sigma \in C^\infty(X, \mathbb{R}) $
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