Gluing variational bicomplexes and homology of multivalued action functionals

XI OPORTO MEETING

on

GEOMETRY, TOPOLOGY and PHYSICS

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References

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Motivation

The Lagrangian is not an n-form but a degree n cocycle

Topological terms

The naive lagrangian is an $(n+1)$-form and the space is n-dimensional

- Group valued maps (G compact Lie group, WZW term)
- Chern-Simons (secondary characteristic classes)

Geometric actions

Action functionals arising from geometric problems:

- Morse theory for multivalued functions and closed 1-forms
- Uniformization of Riemann surfaces — Liouville equation
- (Universal) Projective structure for a Teichmüller curve

Variational Principle

- Smooth submersion $\pi : E \to U$, infinite jet bundle $\pi^\infty : J^\infty E \to U$.
- Coordinates on $J^\infty E$: (x^i, u^α_I)

  $i = 1, \ldots, n$, $\alpha = 1, \ldots, m$, I is a multi-index (bookeeping for derivatives!)
- Local forms on $J^\infty E$:
  - Forms on $J^\infty E$ split thanks to the contact ideal generated by $\theta^\alpha_I = du^\alpha_I - u^\alpha_{I \cup \{i\}}\, dx^i$:

$$\underline{A}^k_{JE} = \bigoplus_{p+q=k} \underline{A}^{p,q}_{JE} \qquad p \text{ number of } \theta^\alpha_I\text{'s}$$

  - Equivalently for $V \subseteq U$ and $\mathcal{S} = \Gamma(V, E)$ use $Ev : \mathcal{S} \times V \xrightarrow{ev} E \xrightarrow{j^\infty} J^\infty E$

$$A^{p,q}_{\text{loc}}(\mathcal{S} \times V) = Ev^* \Big(\pi^\infty_* (\underline{A}^{p,q}_{JE})(V) \Big)$$

The differential splits in the standard fashion

$$d_{J^\infty E} = \underbrace{d_V}_{\theta^\theta_I} + \underbrace{d_H}_{dx^i} = \underbrace{\boldsymbol{\delta}}_{\mathcal{S}} + \underbrace{(-1)^p d}_{M}$$

Variational Principle

- Lagrangian n-form: $\omega = L[x^i, u^\alpha_I]\, dx^1 \wedge \cdots \wedge dx^n$

  local form of degree $(0, n)$ on $J^\infty E$

- Action functional:

$$V \subseteq U\,, \quad s \in \Gamma(V, E)\,, \quad A[s] = \int_V (j^\infty(s))^*(\omega)\,, \quad s \text{ compactly supported}$$

- Variation

$$\boxed{\boldsymbol{\delta}\omega = a + d\gamma}$$

a: source form

γ: Cartan form

- Universal conserved current

$$\boxed{k = \boldsymbol{\delta}\gamma}$$

What is a multivalued action?

Heuristic definition

- $M \equiv M^n$ compact n-dimensional manifold
- $\mathfrak{U}_M = \{U_a\}_{a\in A}$ an open cover.
- A collection of Lagrangians $\omega_a^{(0)} = L_a \, dx_a^1 \wedge \cdots \wedge dx_a^n$.

  $\omega_a^{(0)} \in \pi_*^\infty \underline{A}^{0,n}_{J^\infty E}(U_a)$: the "local Lagrangian" L_a depends on a section of some fiber bundle $E \xrightarrow{\pi} M$.

  This can be generalized

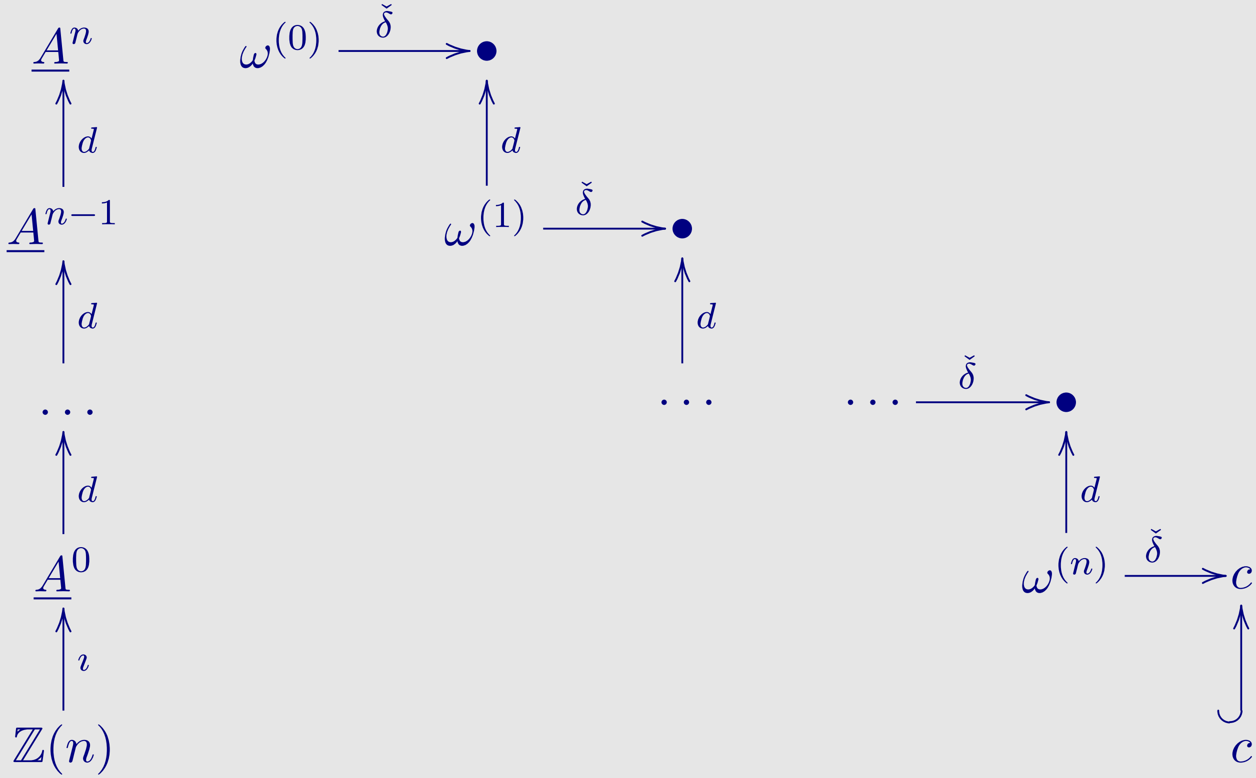
Must compare two local Lagrangian densities $\omega_a^{(0)}$ and $\omega_b^{(0)}$ on $U_{ab} = U_a \cap U_b$.

Main assumption

$$\omega_b^{(0)} - \omega_a^{(0)} = d\omega_{ab}^{(1)} \,, \quad \omega_{ab}^{(1)} \in \pi_*^\infty \underline{A}^{0,n-1}_{J^\infty E}(U_{ab})$$

Descent

The datum $\omega^{(0)}_a$, $\omega^{(1)}_{ab}$ is completed according to the standard *descent staircase:*



- each $\omega^{(k)}_{a_0 a_1 \dots a_k}$ is a local form on a k-fold intersection.
- $c \in \mathbb{Z}(n) \equiv (2\pi\sqrt{-1})^n \mathbb{Z}$ is an "integer."
- The augmentation by $\mathbb{Z}(n)$ is due to "experimental evidence."

A toy example

Let $M = M^3$:

ℂ-line bundle with connection

$$\big(L \longrightarrow M\,,\ \nabla\big)$$

Cocycle description

$$\big(\xi_a, f_{ab}, C_{abc}\big) \quad \begin{cases} \xi_b - \xi_a = -df_{ab} \\ f_{ab} - f_{ac} + f_{bc} = C_{abc} \end{cases}$$

$\omega^{(0)}_a$	$\omega^{(1)}_{ab}$	$\omega^{(2)}_{abc}$	$\omega^{(3)}_{abcd}$	$\omega^{(4)}_{abcde}$
$\xi_a \wedge d\xi_a$	$f_{ab}\, d\xi_b$	$C_{abc}\, \xi_c$	$C_{abc}\, f_{cd}$	$C_{abc}\, C_{cde}$

Variation:

$$\underbrace{\boxed{\boldsymbol{\delta}\xi_a = \boldsymbol{\delta}\xi_b}}_{\text{variations glue}} \quad \Longrightarrow \quad \begin{aligned} \boldsymbol{\delta}\omega^{(0)}_a &= 2\, d\xi_a \wedge \boldsymbol{\delta}\xi_a - d\big(\xi_a \wedge \boldsymbol{\delta}\xi_a\big) \\ \boldsymbol{\delta}\omega^{(1)}_{ab} &= \check{\delta}\big(\xi_\bullet \wedge \boldsymbol{\delta}\xi_\bullet\big)_{ab} + d\big(f_{ab}\, \boldsymbol{\delta}\xi_b\big) \\ \boldsymbol{\delta}\omega^{(2)}_{abc} &= \check{\delta}\big(f_{\bullet\bullet}\, \boldsymbol{\delta}\xi_\bullet\big)_{abc} \end{aligned}$$

Note:

- Class of $\big(\omega^{(0)}_a\,, \omega^{(1)}_{ab}\,, \omega^{(2)}_{abc}\,, \omega^{(3)}_{abcd}\,, \omega^{(4)}_{abcde}\big) = [L\,, \nabla] \cup [L\,, \nabla]$
- $d\omega^{(0)}_a = d\xi_a \wedge d\xi_a$

must use Deligne cohomology!

Would be a well defined 4-form!

Definition of multivalued action

Unfortunately (or more interestingly), we will have to consider *triple* complexes:

For each p — variational index — consider

$$\boxed{\check{C}^\bullet(\mathcal{U}_M, \mathbb{Z}(n) \to \underline{A}^0_M \to \cdots \to \underline{A}^n_M) \longrightarrow \check{C}^\bullet(\mathcal{U}_M, \mathbb{Z}(n) \to (\pi^\infty_* \underline{A}^{p,0}_{J^\infty E} \to \cdots \to \pi^\infty_* \underline{A}^{p,n}_{J^\infty E}))}$$

$$D = d \pm \check{\delta} \qquad \triangle = \boldsymbol{\delta} \pm d + \pm\check{\delta}$$

Definition 1. A multivalued Lagrangian cocycle is a cocycle

$$\Omega = \omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(n)} + c$$

of total degree $n+1$ in the total simple complex associated to $\mathsf{C}^{0,q,r}$. The homogeneous members satisfy the descent condition. If Σ represents the fundamental class $[M]$ of M, the multivalued action functional associated to Ω is given by the evaluation

$$S = \langle \Omega, \Sigma \rangle \equiv \langle [\Omega], [M] \rangle$$

Dynamical fields?

Examples of dynamical fields

Example 1. Cocycles of degree p in some Deligne complex.

$$\boxed{\Phi = \phi^{(0)} + \cdots + \phi^{(p-1)} + c} \qquad \phi^{(j)} \in \check{C}^{p-j}(\mathbb{Z}(p)^j_{\mathcal{D}}) , \quad c = \phi^{(p)} \in \mathbb{Z}(p)$$

dynamical field	gluing law
$\{\phi^{(0)}_a\}_{a\in A}$	$\phi^{(0)}_b - \phi^{(0)}_a = \pm d\phi^{(1)}_{ab}$ $\phi^{(1)}_{ab} - \phi^{(1)}_{ac} + \phi^{(1)}_{bc} = \pm d\phi^{(2)}_{abc}$ $\ldots$

Example 2. Connections on a principal G-bundle over M with non-abelian structure group G and Lie algebra $\mathfrak{g}$.

dynamical field	gluing law
$\{A_a\}_{a\in A}$	$A_b - \mathrm{ad}(g_{ab}^{-1})(A_a) = g_{ab}^{-1} dg_{ab}$

Example 3. (G, X)-structures on M: actions $\underline{G}_M \times \underline{X}_M \to \underline{X}_M$

dynamical field	gluing law
$x_a \in \underline{X}_M(U_a)$	$x_a = g_{ab}(x_b)$

Assumptions on dynamical fields

Definition 2. Consider the data:

- $\mathcal{U}_M \to M$ a covering of M
- $\underline{E}_M$ a sheaf over M with an appropriate structure
- $\{\phi_a \in \underline{E}_M(U_a)\}_{a \in A}$ be a collection of sections

Assume that either:

1. $\underline{E}_M$ can be realized as the highest degree object in a complex of abelian groups

$$\cdots \longrightarrow \underline{E}_M^{-2} \longrightarrow \underline{E}_M^{-1} \longrightarrow \underline{E}_M^{0} \equiv \underline{E}_M$$

2. $\underline{E}_M$ can be realized as the zero level of a truncated simplicial object:

$$\cdots \substack{\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow\\[-4pt]\textstyle\longrightarrow} \underline{E}_M^{-1} \rightrightarrows \underline{E}_M^{0} \equiv \underline{E}_M$$

The descent condition is satisfied for the relevant sheaves of jets (of isomorphisms):

$$\psi_{ab} \in \underline{E}_M^{-1}(U_{ab}) \rightsquigarrow j^\infty(\psi_{ab}) : J^\infty \underline{E}_M|_{U_{ab}} \to J^\infty \underline{E}_M|_{U_{ab}}$$
$$j^\infty(\psi_{ab}) \circ j^\infty(\psi_{bc}) \Longrightarrow j^\infty(\psi_{ac})$$

Situation

- The assumption about dynamical fields implies a gluing property with respect to maps

$$A^{p,q}_{\mathrm{loc}}(\mathcal{S}_b \times U_{ab}) \to A^{p,q}_{\mathrm{loc}}(\mathcal{S}_a \times U_{ab})$$

Lemma 1 (Gluing Lemma). *For $p \geq 1$ the variational complexes $A^{p,\bullet}_{\mathrm{loc}}(\mathcal{S}_a \times U_a)$ descend to a global object on M. Also, $\boldsymbol{\delta}\check{\delta} = \check{\delta}\boldsymbol{\delta}$.*

- *Local* variation

$$\boxed{\boldsymbol{\delta}\omega_a^{(0)} = a_a^{(0)} + d\gamma_a^{(0)}}$$

- Local locus $\mathcal{M}_a \subset \mathcal{S}_a$ defined by the Euler-Lagrange equation $a_a^{(0)} = 0$.
- Local Cartan form and Universal current $k_a^{(0)} = \boldsymbol{\delta}\gamma_a^{(0)}$.

Main results

"Everything glues..."

Theorem 1. *The cochain* $\{a_a^{(0)}\}_{a\in A}$ *is a* 0*-cocycle, it determines a globally defined* $(1,n)$*-source form* $a^{(0)}$*. The variation of the total cocycle* $\Omega = \omega^{(0)} + \omega^{(1)} + \cdots + \omega^{(n)} + c$ *is solely due to the source form up to a total coboundary, namely*

$$\boldsymbol{\delta}\Omega = a^{(0)} + D\Gamma\,,$$

where $\Gamma \equiv \sum_{q=0}^{n-1} \gamma^{(q)}$ *is a chain of total degree* n *in the such the last component* $\gamma^{(n)} = 0$*.*

Corollary 2. *The zero-loci* $\mathcal{M}_a \subset \mathcal{S}_a$ *glue into a global locus* $\mathcal{M}$*.*

There is a global Universal current

$$\boxed{K \stackrel{\text{def}}{=} \boldsymbol{\delta}\Gamma}$$

Proposition 3. *The global current* K *satisfies*

$$\boldsymbol{\delta}K = 0\,, \qquad D\,K = -\boldsymbol{\delta}a^{(0)}$$

so it is a conserved current. The restriction to $\mathcal{M}$ *is closed with respect to* $\boldsymbol{\delta} + D$*.*

A real-life example

$$\begin{array}{rcl} \text{Universal Projective Structure} \quad \mathcal{P}(X) & = & \{(h,\mu) | \mathcal{D}_h \mu - \bar{\partial} h = 0\} / \mathcal{G}(X)_0 \\ \downarrow & & \\ \text{Teichmüller Space} \quad \mathcal{T}(X) & = & \{[f : X \to \tilde{X}]\} \end{array}$$

$f : X \to \tilde{X}$: quasi-conformal map. $\mu = \bar{\partial} f / \partial f \in A^{0,1}(X, T_X)$: Beltrami differential.

h: projective connection on X. $\mathcal{D}_h = \partial^3 + 2\, h\, \partial + \partial h$.

Variation: $\boldsymbol{\delta} f_a = w'_{ab}(f_b)\, \boldsymbol{\delta} f_b + \boldsymbol{\delta} w_{ab}$

$$\underbrace{\boxed{\boldsymbol{\delta} f_a = w'_{ab}(f_b)\, \boldsymbol{\delta} f_b}}_{\text{variations glue}} \Leftrightarrow KS \equiv [\boldsymbol{\delta} w_{ab}] = 0 \Leftrightarrow \boxed{\text{Vertical variations}}$$

$\boxed{\mathcal{D}_h \mu - \bar{\partial} h = 0}$: Euler-Lagrange for

$$\boxed{\begin{array}{c} \omega_a^{(0)} = \dfrac{\partial^2 f_a}{\partial f_a}\, \partial \mu_a\, dz_a \wedge d\bar{z}_a + 2\, \mu_a\, h_a\, dz_a \wedge d\bar{z}_a \\ \omega_{ab}^{(1)}, \quad \omega_{abc}^{(2)} \rightsquigarrow \big(T_X, T_X\big] \end{array}}$$

Conjectural generalization

$$\begin{array}{ccccccc}
\cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \underline{E}_M^{-2} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \underline{E}_M^{-1} & \substack{\longrightarrow\\ \longrightarrow} & \underline{E}_M^{0} \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathcal{U} \times_M \mathcal{U} \times_M \mathcal{U} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathcal{U} \times_M \mathcal{U} & \substack{\longrightarrow\\ \longrightarrow} & \mathcal{U}
\end{array}$$

A form of the Theorem should be valid with a cocycle a of length equal to the number of members in the complexes above that are allowed to have dynamical fields