

**Hermitian-holomorphic classes and tame symbols related to
uniformization, the dilogarithm, and the Liouville Action**

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- Uniformization

Constant ($= -1$) negative curvature metrics on Riemann surfaces

- Liouville Action

- Characteristic Classes

- (Hermitian-Holomorphic) Deligne cohomology

- Determinant of cohomology

- Regulator class, aka hyperbolic 3-volume

Riemann Surfaces and constant ($= -1$) negative curvature metrics

X compact Riemann surface, genus ≥ 2 .

Conformal metric:

$$ds^2 = \rho dz \otimes d\bar{z}, \quad \rho \in C^\infty(U, \mathbb{R}_{>0}), \quad U \subset X \text{ open.}$$

Line bundle with hermitian metric: (\underline{T}_X, ρ)

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Line bundle with hermitian metric: (\underline{T}_X, ρ)

Scalar curvature $K_\rho = -1$ iff

$$\frac{\partial^2}{\partial z \partial \bar{z}} \log \rho = \frac{1}{2} \rho$$

Liouville equation

Classical theory after Liouville, Picard, Poincaré,...

The Liouville equation admits a local variational principle

Let $D \subset \mathbb{C}$ be a region and $\rho \in C^\infty(D, \mathbb{R}_{>0})$. Set:

$$S_D[\phi] = \frac{\sqrt{-1}}{2} \int_D \left(\partial\phi \wedge \bar{\partial}\phi + e^\phi dz \wedge d\bar{z} \right), \quad \phi = \log \rho$$

Then with $\sigma \in C^\infty(D, \mathbb{R})$ we have:

$$K_\rho = -1 \iff dS_D[\phi](\sigma) \equiv \left. \frac{d}{dt} \right|_{t=0} S_D[\phi + t\sigma] = 0$$

Definition 1. The functional $S_D[\phi]$ is the **Liouville Action** attached to the region $D \subset \mathbb{C}$.

One problem...

$\omega^{(0)}[\phi] \stackrel{\text{def}}{=} \partial\phi \wedge \bar{\partial}\phi$ does not glue as a global 2-form.

$f : D' \longrightarrow D$, analytic change of coordinates. Denote by ϕ_D and $\phi_{D'}$ the Liouville fields on D and D' .

The Liouville equation is invariant under f if and only if

$$\phi_{D'}(z) - f^*\phi_D(z) = \log |f'(z)|^2$$

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As a consequence:

$$f^*(\partial\phi_D \wedge \bar{\partial}\phi_D) \neq \partial\phi_{D'} \wedge \bar{\partial}\phi_{D'}$$

$\check{S}_D[\phi] \stackrel{\text{def}}{=} \int_D \omega^{(0)}$ is ill-defined: $\check{S}_D[\phi_D] \equiv \check{S}_{D'}[f^*\phi_D] \neq \check{S}_{D'}[\phi_{D'}]$

\implies No extension of the action to X !

Liouville action via Descent

(L.Takhtajan-P.Zograf, E.A.-L.Takhtajan, L.Takhtajan-L.-P.Teo, E.A.)

- Choose a cover $\mathfrak{U}_X = \{U_i\}_{i \in I} \rightarrow X$ and set $\omega^{(0)} = \{\omega_i^{(0)}[\phi_i]\}$
- Find $\omega_{ij}^{(1)}$ and $\omega_{ijk}^{(2)}$ to complete the **Descent staircase**:

$$\begin{array}{c}
 \underline{\mathcal{E}}_X^2 \otimes_{\mathbb{R}} \mathbb{R}(1) \\
 \uparrow d \\
 \underline{\mathcal{E}}_X^1 \otimes_{\mathbb{R}} \mathbb{R}(1) \\
 \uparrow d \\
 \underline{\mathcal{E}}_X^0 \otimes_{\mathbb{R}} \mathbb{R}(1)
 \end{array}$$

$$\begin{array}{c}
 \omega^{(0)} \xrightarrow{\delta} \bullet \\
 \uparrow d \\
 \omega^{(1)} \xrightarrow{\delta} \bullet \\
 \uparrow d \\
 \omega^{(2)} \xrightarrow{\delta} 0
 \end{array}$$

- Define $\Omega[\phi] \stackrel{\text{def}}{=} \omega^{(0)} + \omega^{(1)} + \omega^{(2)}$ and set:

$$\check{S}_X[\phi] = -\frac{1}{2\pi\sqrt{-1}} \langle \Omega[\phi], [X] \rangle$$

furthermore define the **Liouville action for X**

$$S_X[\phi] = \check{S}_X[\phi] + \frac{1}{2\pi} \int_X \text{vol}_\rho$$

- Finally, prove

Theorem 1. *For any $\sigma \in C^\infty(X, \mathbb{R})$ we have*

$$K_\rho = -1 \iff \left. \frac{d}{dt} \right|_{t=0} S_X[\phi + t\sigma] = 0$$

Question: Why do the descent equations close?

Let $g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$ be the transition functions of the tangent line sheaf \underline{T}_X . One computes

$$\delta\omega^{(1)}_{ijk} = g_{ij} \cup g_{jk} \quad (\text{Real Deligne cohomology})$$

In fact one has

$$g_{ij} \cup g_{jk} = \text{Image of } (g_{ij}, g_{jk})$$

(Tame symbol in \mathbb{Z} -Deligne cohomology)

The symbols (g_{ij}, g_{jk}) glue into a (necessarily neutral i.e. trivial) **2-Stack** $(\underline{T}_X, \underline{T}_X)$ on X .

Consequence: there exist local Bloch-Wigner dilogarithms \mathcal{L}_{ijk} such that

$$d\omega_{ijk}^{(2)} \equiv d\mathcal{L}_{ijk} = g_{ij} \cup g_{jk}, \quad \delta\mathcal{L}_{ijkl} = 0$$

A crash course in Deligne cohomology

- Subring $A \subset \mathbb{R}$. Set $A(p) = (2\pi\sqrt{-1})^p A$.
- **Deligne complex** (X any complex manifold)

$$\begin{aligned} A(p)_{\mathcal{D}}^{\bullet} &= A(p)_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{d} \underline{\Omega}_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \underline{\Omega}_X^{p-1} \\ &\xrightarrow{\cong} \text{Cone} \left(A(p)_X \oplus F^p \underline{\Omega}_X^{\bullet} \xrightarrow{i-j} \underline{\Omega}_X^{\bullet} \right) [-1] \end{aligned}$$

- **Deligne Cohomology:** $H_{\mathcal{D}}^{\bullet}(X, A(p)) = \mathbf{H}^{\bullet}(X, A(p)_{\mathcal{D}}^{\bullet})$
- **Beilinson cup-product:** a family

$$\cup_{\alpha} : A(p)_{\mathcal{D}}^{\bullet} \otimes A(q)_{\mathcal{D}}^{\bullet} \longrightarrow A(p+q)_{\mathcal{D}}^{\bullet}$$

induces a graded-commutative cup-product (**Tame symbol map**)

$$H_{\mathcal{D}}^i(X, A(p)) \otimes H_{\mathcal{D}}^j(X, A(q)) \xrightarrow{\cup} H_{\mathcal{D}}^{i+j}(X, A(p+q))$$

- $\mathbb{Z}(1)_{\mathcal{D}}^{\bullet} \xrightarrow{\cong} \mathcal{O}_X^{\times}[-1] \Rightarrow H_{\mathcal{D}}^{\bullet}(X, \mathbb{Z}(1)) \cong H^{\bullet-1}(X, \mathcal{O}_X^{\times})$
 - $\mathcal{O}_X^{\times}(U) \cong H_{\mathcal{D}}^1(U, \mathbb{Z}(1))$ $f \sim (\log_i f, 2\pi\sqrt{-1} n_{ij})$
 - $\text{Pic } X \cong H_{\mathcal{D}}^2(X, \mathbb{Z}(1))$ $L \sim (\log g_{ij}, 2\pi\sqrt{-1} c_{ijk})$
- $H_{\mathcal{D}}^1(X, \mathbb{R}(1))$ $f \sim (d \log f, \log |f|)$
- $\mathbb{Z}(2)_{\mathcal{D}}^{\bullet} \xrightarrow{\cong} (\mathcal{O}_X^{\times} \rightarrow \underline{\Omega}_X^1)[-1]$
 - $H_{\mathcal{D}}^2(X, \mathbb{Z}(2)) \cong$ Isomorphism classes of invertible sheaves with connection
- Beilinson product & tame symbol

$$H_{\mathcal{D}}^1(U, \mathbb{Z}(1)) \otimes H_{\mathcal{D}}^1(U, \mathbb{Z}(1)) \ni f \otimes g \mapsto (f, g] \in H_{\mathcal{D}}^2(U, \mathbb{Z}(2))$$

$$(f, g] \sim (g^{n_{ij}}, \log_i f d \log g) \quad (\text{Deligne})$$

$$H_{\mathcal{D}}^2(X, \mathbb{Z}(1)) \otimes H_{\mathcal{D}}^2(X, \mathbb{Z}(1)) \ni L \otimes L' \mapsto (L, L'] \in H_{\mathcal{D}}^4(X, \mathbb{Z}(2))$$

$$(L, L'] \sim (g'_{kl}{}^{c_{ijk}}, \log g_{ij} d \log g'_{jk}) \quad (\text{Brylinski-McLaughlin})$$

Hermitian metrics

Enhanced cohomology to include hermitian structures

- $L_\rho = (L, \rho)$ Invertible sheaf on X with hermitian fiber metric.
Isomorphism classes: $\widehat{\text{Pic}} X$
- Deligne's observation: $\widehat{\text{Pic}} X \cong \mathbf{H}^2(X, \mathbb{Z}(1) \xrightarrow{i} \mathcal{O}_X \xrightarrow{Re} \underline{\mathcal{E}}_X^0)$
- In general: Hermitian-holomorphic Deligne cohomology

$$D(p)_{h.h.}^\bullet = \text{Cone}(\mathbb{Z}(p)_{\mathcal{D}}^\bullet \oplus (F^p \underline{A}_X^\bullet \cap \sigma^{2p} \underline{\mathcal{E}}_X^\bullet(p)) \longrightarrow \mathbb{R}(p)_{\mathcal{D}}^\bullet)[-1]$$

$$H_{\mathcal{D}_{h.h.}}^\bullet(X, p) := \mathbf{H}^\bullet(X, D(p)_{h.h.}^\bullet)$$

- Quasi-isomorphism: $D(1)_{h.h.}^\bullet \xrightarrow{\cong} \mathbb{Z}(1) \xrightarrow{i} \mathcal{O}_X \xrightarrow{Re} \underline{\mathcal{E}}_X^0$

$$\text{Thus: } \widehat{\text{Pic}} X \cong H_{\mathcal{D}_{h.h.}}^2(X, 1)$$

Cup product for Hermitian holomorphic Deligne cohomology

Since $D(p)_{h.h.}^\bullet = \text{Cone}(\dots)$ there is a **Beilinson cup-product**:

$$H_{\mathcal{D}_{h.h.}}^i(X, p) \otimes H_{\mathcal{D}_{h.h.}}^j(X, q) \xrightarrow{\cup} H_{\mathcal{D}_{h.h.}}^{i+j}(X, p+q)$$

In particular for $L_\rho, L'_{\rho'} \in \widehat{\text{Pic}} X$ we have

$$\begin{array}{ccc} L_\rho \otimes L'_{\rho'} & \longrightarrow & (L_\rho, L'_{\rho'}) \in H_{\mathcal{D}_{h.h.}}^4(X, 2) \\ \downarrow & & \downarrow \text{forget} \\ L \otimes L' & \longrightarrow & (L, L') \in H_{\mathcal{D}}^4(X, \mathbb{Z}(2)) \\ & & \downarrow \\ & & \underline{2\text{-STACK}}(X) \end{array}$$

We want to apply this to the case $L = L' = \underline{T}_X$ on a curve X with conformal metric ρ

On a curve

Back to a compact Riemann surface X (genus ≥ 2) with conformal metric ρ . We have $\underline{T}_{X,\rho} \in \widehat{\text{Pic}} X$.

- Set: $\check{S}_X[\rho] := -(2\pi\sqrt{-1})^{-1} \otimes (\underline{T}_{X,\rho}, \underline{T}_{X,\rho}]$
- For dimensional reasons $\cdots \rightarrow H_{\mathcal{D}}^3(X, \mathbb{R}(2)) \rightarrow H_{\mathcal{D}_{h.h.}}^4(X, 2) \rightarrow 0$ and $H_{\mathcal{D}}^3(X, \mathbb{R}(2)) \cong H^2(X, \mathbb{R}(1))$ (since $\mathbb{R}(2)_{\mathcal{D}}^{\bullet} \xrightarrow{\cong} \underline{\mathcal{E}}_X^{\bullet}(1)[-1]$.) Indeed:

$$\begin{array}{ccccc}
 \mathbb{Z}(2) & \xrightarrow{-\iota} & \mathcal{O}_X & \xrightarrow{-d} & \underline{\Omega}_X^1 \\
 D(2)_{h.h.}^{\bullet} = & & \downarrow \pi_1 & & \downarrow \pi_1 \\
 & & \underline{\mathcal{E}}_X^0 \otimes_{\mathbb{R}} \mathbb{R}(1) & \xrightarrow{-d} & \underline{\mathcal{E}}_X^1 \otimes_{\mathbb{R}} \mathbb{R}(1) & \xrightarrow{-d} & \underline{\mathcal{E}}_X^2 \otimes_{\mathbb{R}} \mathbb{R}(1)
 \end{array}$$

On a curve X $\check{S}_X[\rho]$ is an \mathbb{R} valued class of degree 2

Geometric functional

Theorem 2. Set $S_X[\rho] := \check{S}_X[\rho] + \frac{1}{2\pi} \int_X \text{vol}_\rho$ We have:

$$K_\rho = -1 \iff \left. \frac{d}{dt} \right|_{t=0} S_X[e^{t\sigma} \rho] = 0 \quad \forall \sigma \in C^\infty(X, \mathbb{R})$$

- Choose a cover $\mathfrak{U}_X \rightarrow X$
- A cocycle $\Omega_{\mathfrak{U}_X}[\rho]$ representing the cup-product $(\underline{T}_{X_\rho}, \underline{T}_{X_\rho})$ can be explicitly computed and it coincides (up to Čech coboundary) with the cocycle constructed from **Descent theory**
- One verifies

$$\left. \frac{d}{dt} \right|_{t=0} \Omega_{\mathfrak{U}_X}[e^{t\sigma} \rho] - \sqrt{-1} \text{vol}_{e^{t\sigma} \rho} \equiv \sigma(K_\rho + 1) \text{vol}_\rho,$$

where \equiv means “up to total coboundary.”

Deligne's determinant of cohomology

- To a pair $L, L' \in \text{Pic } X$ is assigned a \mathbb{C} -vector space $\langle L, L' \rangle$.
 - s and s' : rational sections with disjoint divisors D and D' , respectively. $\langle s, s' \rangle$ is a generator $/\mathbb{C}$
 - f and g rational functions. Relations:

$$\langle fs, s' \rangle \equiv f(\text{div}(s'))\langle s, s' \rangle \quad \langle s, gs' \rangle \equiv g(\text{div}(s))\langle s, s' \rangle$$

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$$\langle fs, s' \rangle \equiv f(\text{div}(s')) \langle s, s' \rangle \quad \langle s, gs' \rangle \equiv g(\text{div}(s)) \langle s, s' \rangle$$

- If $L_\rho, L'_{\rho'} \in \widehat{\text{Pic } X}$ (denote the metric for both bundles by $\|\cdot\|$)

$$\log \|\langle s, s' \rangle\|^2 = \frac{1}{2\pi\sqrt{-1}} \int_X \partial\bar{\partial} \log\|s\|^2 \log\|s'\|^2 + \log\|s\|^2[D'] + \log\|s'\|^2[D]$$

defines a **Hermitian metric** on the \mathbb{C} vector space $\langle L, L' \rangle$: the **Determinant of cohomology**

Determinant of cohomology and Hermitian-holomorphic Deligne cohomology

In general, we have

Theorem 3. *The cup product of $(L_\rho, L'_{\rho'}) \in H_{\mathcal{D}_{h.h.}}^4(X, 2)$ in hermitian holomorphic Deligne cohomology corresponds to the norm $\|\cdot\|$ on the Deligne pairing $\langle L, L' \rangle$. The proportionality factor is $-\pi \sqrt{-1}$.*

In particular, for $L_\rho = L'_{\rho'} = \underline{T}_X$ we have obtained

Corollary 4. *The Liouville action without area $\check{S}_X[\rho]$ is a cup-square. The exponential of the Liouville action defines a hermitian metric on the complex line $\langle \underline{T}_X, \underline{T}_X \rangle$, namely:*

$$\exp S_X[\rho] = \|\langle \underline{T}_X, \underline{T}_X \rangle\| \exp \frac{1}{2\pi} \int_X \text{vol}_\rho.$$

Hyperbolic volume classes and holography

$\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ *geometrically finite* purely loxodromic second kind Kleinian group

- *Discontinuity set* $\Omega_\Gamma \subset \mathbb{P}^1$
- *Riemann surface* $X = \Omega_\Gamma / \Gamma \cong X_1 \sqcup \cdots \sqcup X_n$
- *Various associated 3-manifolds:*

$$M = \mathbb{H}^3 / \Gamma, \quad N = (\mathbb{H}^3 \cup \Omega_\Gamma) / \Gamma, \quad X = \partial N$$

Krasnov (Schottky), L. Takhtajan-L.-P. Teo (Class A Kleinian) proved

Theorem 5. $\check{S}_X[\rho]$ equals (up to an explicit constant) $\mathrm{vol}^{\mathrm{reg}}(M)$

Characteristic class interpretation?

Classical dilogarithm(s)

- Euler's dilogarithm

$$Li_2(z) = - \int^z \log(1-t) \frac{dt}{t}, \quad z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

- (Logarithm of a) Flat trivializing section of $(1-z, z]$

- Bloch-Wigner dilogarithm (Single valued!)

$$\mathcal{D}_2(z) = \arg(1-z) \log |z| + \operatorname{Im} Li_2(z), \quad z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

- (Logarithm of) Hermitian square of $Li_2(z)$

- Hyperbolic volume: $\mathcal{D}_2(z) = \operatorname{Vol}_{\mathbb{H}^3}(\Delta(\infty, 0, 1, z))$ (Volume of the hyperbolic ideal tetrahedron)

- Regulator class: $H^3(\operatorname{PSL}_2(\mathbb{C})^\delta, \mathbb{R}(1)) \cong \mathbb{R}[\sqrt{-1}\mathcal{D}_2]$

In fact: $[\sqrt{-1}\mathcal{D}_2] = 2 \operatorname{Im} CS$, $CS \in \hat{H}^3(B\operatorname{PSL}_2(\mathbb{C})^\delta, \mathbb{C}/\mathbb{Z})$

A transgression Since $M = \mathbb{H}^3/\Gamma$ is non-compact, the pullback of the regulator to $B\Gamma$ vanishes:

$$H^3(B\Gamma, \mathbb{R}(1)) \longleftarrow^{Bi^*} H^3(B\mathrm{PSL}_2(\mathbb{C}), \mathbb{R}(1))$$

$$0 \longleftarrow \longrightarrow [\sqrt{-1}\mathcal{D}_2]$$

Theorem 6. *The class $i^*[\sqrt{-1}\mathcal{D}_2]$ is the **obstruction** to closing the descent equations (i.e. calculating a cocycle) for $(\underline{T}_{X_\rho}, \underline{T}_{X_\rho})$ with respect to the étale cover $\Omega_\Gamma \rightarrow X$. This obstruction is zero for Γ of the second kind.*

Can be interpreted as: $\mathrm{Transgr}([\sqrt{-1}\mathcal{D}_2]) = (\underline{T}_{X_\rho}, \underline{T}_{X_\rho})$ for

$$\mathrm{hom} - \mathrm{fiber} \rightarrow B\Gamma \rightarrow B\mathrm{PSL}_2(\mathbb{C})$$

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- Is there a similar structure in $5 = 4 + 1$ i.e. \mathbb{H}^5 and manifolds of the form \mathbb{H}^5/Γ and their ideal 4-dim boundaries? If yes, does it involve the **Trilogarithm**?