CONVERGENCE OF A MONTE CARLO METHOD FOR FULLY NON-LINEAR ELLIPTIC AND PARABOLIC PDES IN SOME GENERAL DOMAINS

ARASH FAHIM

ABSTRACT. In this paper, we introduce a probabilistic numerical scheme for a class of parabolic and elliptic fully non-linear PDEs in bounded domains. In the main result, we provide the convergence of a discrete-time approximation to the viscosity solution of a fully non-linear parabolic equation by assuming that comparison principle holds for the PDE.

1. INTRODUCTION

The aim of this paper is to introduce a probabilistic numerical method for a class of elliptic and parabolic fully non-linear boundary value problems on bounded domains. We follow the same methodology as in [7] and [8]. The proof of convergence relies on the result of [1] where the convergence of general monotone schemes are discussed. Since the domain is bounded, we introduce a discrete exit time of the Euler discretization of the underlying process.

The Monte Carlo approximation is provided for semi-linear PDEs on \( \mathbb{R}^d \) in [12] and [5], for semi-linear PDEs with free boundary in [11] and [3], and for semi-linear PDEs in bounded regular domains [3]. In all these papers backward stochastic differential equations (BSDE) is used to provide the approximation. If the domain is \( \mathbb{R}^d \), the rate of convergence obtained in [12] and [3] is \( h^{1/2} \) where \( h \) is the length time step. For free boundary problems, \( h^{1/2} \) is obtained for the rate of convergence in [11] and [3]. For bounded regular domains, in [3] the rate of \( h^{1/4} - \varepsilon \) is provided for any \( \varepsilon > 0 \). In all the mentioned results are obtained by using the regularity of the solution of BSDE. For fully non-linear PDEs on \( \mathbb{R}^d \), a Monte Carlo approximation is obtained in [7] without using BSDEs. By using the methodology of [7], [3] provide a Monte Carlo approximation for fully non-linear free boundary problems.

The paper is organized as follows. In Section 2, we present the numerical scheme. In Section 3, the main result is explained. Finally, Section 4 provides the proofs of the main results.

Notations. The collection of all symmetric \( d \times d \) matrices is denoted \( \mathbb{S}_d \), and its subset of non-negative symmetric matrices is denoted by \( \mathbb{S}_d^+ \). For \( A \in \mathbb{S}_d^+ \), \( A^- \) is the pseudo-inverse of matrix \( A \). We consider an \( \mathbb{R}^d \)-valued Brownian motion \( W \) on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where the filtration \( \mathbb{F} = \{\mathcal{F}_t, t \in [0, T]\} \) satisfies the usual conditions. Finally, we set \( \mathbb{E}_{t,x} := \mathbb{E}[\cdot | X_t = x] \)

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for a pre-specified diffusion process $X$ and $\mathbb{E}_t := \mathbb{E}[\mathcal{F}_t]$.

2. DISCRETIZATION

Let $\mathcal{O} \subset \mathbb{R}^d$ be a domain with boundary shown by $\partial \mathcal{O}$. We consider the boundary value problem:

\[ -L^X v - F(\cdot, v, Dv, D^2v) = 0, \quad \text{on } [0, T) \times \mathcal{O}, \tag{2.1} \]
\[ v = g, \quad \text{on } ([0, T) \times (\mathbb{R}^d \setminus \mathcal{O})) \cup (\{T\} \times \mathbb{R}^d). \tag{2.2} \]

where

\[ L^X \varphi := \frac{\partial \varphi}{\partial t} + \mu \cdot D\varphi + \frac{1}{2} a \cdot D^2\varphi, \]

is a linear operator,

\[ F : (t, x, r, p, \gamma) \in \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d \mapsto F(x, r, p, \gamma) \in \mathbb{R}, \]

is a non-linear map, $\mu$ and $\sigma$ are maps from $\mathbb{R}_+ \times \mathcal{O}$ to $\mathbb{R}^d$ and $\mathcal{M}(d, d)$, respectively, $a := \sigma^T$, and $g : [0, T) \times (\mathbb{R}^d \setminus \mathcal{O}) \cup \{T\} \times \mathbb{R}^d \to \mathbb{R}$. Observe that for the sake of numerical schemes, we need $g$ to be defined beyond the parabolic boundary of $[0, T) \times \mathcal{O}$.

For a positive integer $n$, let $h := T/n$, $t_i = ih$, $i = 0, \ldots, n$, and consider the one step ahead Euler discretization

\[ \hat{X}_h^{t_i} := x + \mu(t, x)h + \sigma(t, x)(W_{t_i+h} - W_t), \tag{2.3} \]

of the diffusion $X$ corresponding to the linear operator $L^X$. Then, the Euler discretization of the process $X$ is defined by $\hat{X}_{t_{i+1}} := \hat{X}_h^{t_{i+1}}$. We also define the first discrete exit time of $X$ from $\mathcal{O}$ by

\[ \hat{\tau} := \inf \left\{ t_i; \hat{X}_{t_i} \notin \mathcal{O} \right\}. \tag{2.4} \]

and $\hat{X}_t := \hat{X}_{t \wedge \hat{\tau}}$, i.e. $\hat{X}$ stopped at stopping time $\hat{\tau}$.

If we assume that the PDE (2.1) has a classical solution, it follows from Itô’s formula that for $x \in \mathcal{O}$ and $t_i \in [0, T)$, we can write

\[ \mathbb{E}_{t_i,x} \left[ v\left( \hat{\tau} \wedge t_{i+1}, \hat{X}_{t_{i+1}} \right) \right] = v(t_i, x) + \mathbb{E}_{t_i,x} \left[ \int_{t_i}^{\hat{\tau} \wedge t_{i+1}} L^X v(t, \hat{X}_t) dt \right], \]

holds true where we ignored the difficulties related to local martingale part. Since $v$ solves the PDE (2.1), we have

\[ v(t_i, x) = \mathbb{E}_{t_i,x} \left[ v\left( \hat{\tau} \wedge t_{i+1}, \hat{X}_{t_{i+1}} \right) \right] + \mathbb{E}_{t_i,x} \left[ \int_{t_i}^{\hat{\tau} \wedge t_{i+1}} F(\cdot, v, Dv, D^2v)(t, \hat{X}_t) dt \right]. \]

Observe that if $x \in \mathcal{O}$, then $\hat{\tau} \geq t_{i+1}$ and therefore, $\hat{\tau} \wedge t_{i+1} = t_{i+1}$. By approximating the Riemann integral, this suggest the following approximation of the value function $v$

\[ v^h(T, \cdot) := g \quad \text{and} \quad v^h(t, x) := T_h[v^h](t, x) \text{ for any } x \in \mathbb{R}^d, \tag{2.5} \]
where we denoted for a bounded function $\psi : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}$

$$T_h[\psi](t, x) := \begin{cases} \mathbb{E}_{t,x} \left[ \psi(t + h, \tilde{X}_{t+h}) \right] + hF \left( \cdot, D_h\psi \right)(t, x) & \text{if } x \in \mathcal{O}, \\ g(t, x) & \text{if } x \notin \mathcal{O}, \end{cases} \quad (2.6)$$

with

$$D_h\psi(t_i, x) = \mathbb{E}_{t,x} \left[ \psi(t + h, \tilde{X}_{t+h}) H_h(t, x) \right], \quad (2.7)$$

where $H_h = (H^h_0, H^h_1, H^h_2)^T$ and

$$H^h_0 = 1, \quad H^h_1(t, x) = (\sigma^T)^{-1} \frac{W_h}{h}, \quad H^h_2(t, x) = (\sigma^T)^{-1} \frac{W_h W^T_h - h I_d}{h^2} \sigma^{-1}.$$ 

For details on the approximation of the derivatives (2.7) see Lemma 2.1 in [7].

3. Asymptotics of the discrete-time approximation

3.1. The main result. Our first main convergence results follow the general methodology of Barles and Souganidis [1], and therefore requires that the nonlinear PDE (2.1) satisfies a comparison result in the sense of viscosity solutions.

We recall that an upper semi-continuous (resp. lower semi-continuous) function $\underline{v}$ (resp. $\overline{v}$) on $[0, T] \times \overline{\mathcal{O}}$, is called a viscosity sub-solution (resp. super-solution) of (2.1) if for any $(t, x) \in [0, T] \times \overline{\mathcal{O}}$ and any smooth function $\varphi$ satisfying

$$0 = (\underline{v} - \varphi)(t, x) = \max_{[0, T] \times \overline{\mathcal{O}}} (\underline{v} - \varphi) \left( \text{resp. } 0 = (\varphi - \overline{v})(t, x) = \min_{[0, T] \times \overline{\mathcal{O}}} (\varphi - \overline{v}) \right), \quad (3.1)$$

we have:

- if $t < T$ and $x \in \mathcal{O}$
  $$- L^X \varphi - F(t, x, D\varphi(t, x)) \leq (\text{resp. } \geq) \ 0,$$

- if $t < T$ and $x \in \partial \mathcal{O}$
  $$\min \left\{ -L^X \varphi - F(t, x, D\varphi(t, x)), (\varphi - g)(t, x) \right\} \leq 0 \quad (\text{resp. } \max \left\{ -L^X \varphi - F(t, x, D\varphi(t, x)), (\varphi - g)(t, x) \right\} \geq 0),$$

- if $t = T$, $\underline{v} - g \leq 0$ (resp. $\overline{v} - g \geq 0$).

It is worth mentioning that by Remark 3.2 of [7], in the above definition, we treated the boundary condition and terminal condition in different ways and moreover, the comparison principle for (2.1)-(2.2) should be given in the following way:

**Definition 3.1.** We say that (2.1)-(2.2) has a comparison for bounded functions if for any bounded upper semi-continuous sub-solution $\underline{v}$ and any bounded lower semi-continuous super-solution $\overline{v}$ on $[0, T) \times \overline{\mathcal{O}}$, satisfying

$$\underline{v}(T, \cdot) \leq \overline{v}(T, \cdot),$$

we have $\underline{v} \leq \overline{v}$.
We denote by $F_r$, $F_p$ and $F_\gamma$ the partial gradients of $F$ with respect to $r$, $p$ and $\gamma$, respectively.

**Assumption F** (i) The non-linearity $F$ is Lipschitz-continuous with respect to $(x,r,p,\gamma)$ uniformly in $t$, and $|F(\cdot,\cdot,0,0,0)|_\infty < K$ for some positive constant $K$;
(ii) $F$ is elliptic and dominated by the diffusion of the linear operator $L^X$, i.e.
\[
\nabla_\gamma F \leq a \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times S_d;
\]
(iii) $F_p \in \text{Image}(F_\gamma)$ and $|F_p^T F_\gamma F_p|_\infty < K$;
(iv) $F_r - \frac{1}{4} F_p^T F_\gamma F_p \geq 0$.

Assumption F(iv) is only made for the sale of simplicity and can be released; see Lemma 3.19 in [7].

**Theorem 3.2** (Convergence). Let Assumption F hold true, and $|\mu|_1$, $|\sigma|_1 < \infty$ and $\sigma$ is invertible. Also assume that the fully nonlinear PDE (2.1) has comparison for bounded functions. Then for every bounded $g$ Lipschitz function on $x$ and $\frac{1}{2}$-Hölder continuous on $t$, there exists a bounded function $v$ such that $v^h \longrightarrow v$ locally uniformly. In addition, $v$ is the unique bounded viscosity solution of problem (2.1)-(2.2).

### 4. Proofs of the main results

**4.1. Proof of the convergence result.** The proof Theorem 3.2 uses the argument of [1] which introduces the following sufficient conditions for convergence:

(i) Let $\varphi$ be a smooth function with bounded derivatives. Then for all $(t,x) \in [0,T) \times \mathbb{R}^d$:
\[
\lim_{(t',x') \to (t,x)} \frac{|c + \varphi(t',x') - T_h[c + \varphi(t',x')]|}{h} = - (L^X \varphi + F(\cdot,\varphi,D\varphi,D^2\varphi))(t,x). \quad \square
\]

(ii) Let $\varphi, \psi : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$ be two exponentially bounded functions. Then:
\[
\varphi \leq \psi \implies T_h[\varphi](t,x) \leq T_h[\psi](t,x).
\]

(iii) $(v^h)_h$ is $L^\infty$-bounded, uniformly in $h$.
(iv) $\lim_{t' \to T} v^h(t',x') = g(T,x)$.

Throughout this section, all the conditions of Theorem 3.2 are in force. (i-ii) are the result of Lemma 3.11, Lemma 3.12, and Remark 3.13 of [7]. We follow the section by proving (iii) and (iv); Lemma 4.1 and Lemma 4.2.

**Lemma 4.1.** If $g$ is bounded on $\mathbb{R}^d \setminus \mathcal{O}$, the family $(v^h)_h$ defined in (2.2)-(2.4) is bounded on $\bar{\mathcal{O}}$, uniformly in $h$.

**Proof.** By Lipschitz property of $F$, one can write
\[
|v^h(t_i,x)| \leq h |F(t,x,0,0,0)| + C_{i+1}(1 + Ch) \quad \text{if} \quad x \in \mathcal{O}, \quad \text{and} \quad |v^h(t_i,x)| \leq C_n \quad \text{if} \quad x \notin \mathcal{O},
\]
where \( C_n = \sup \{g(t,x)|(t,x) \in [0,T) \times \Omega^j \cup T \times \mathbb{R}^d \} \) and \( C_j = \|u^h(t_j,\cdot)\|_\infty \) for \( j \leq n-1 \); see Lemma 3.14 in [2]. Define \( \tilde{C}_j := \max \{ C_j, C_n \} \). Since \( g \) is bounded, \( \tilde{C}_n = C_n < \infty \). Therefore, we have
\[
\tilde{C}_i \leq hK + \tilde{C}_{i+1}(1 + Ch).
\]

By proceeding backward induction, we obtain \( \tilde{C}_i \leq C_n e^{CT} \) for some constant \( C \) independent of \( h \) which concludes the argument.

Finally, we prove that the terminal condition is preserved by our scheme as the time step shrinks to zero.

**Lemma 4.2.** For each \( x \in \mathcal{O} \) and \( t_i = ih \) with \( i = 1, \ldots, n \), we have;
\[
|v^h(t_i, x) - g(T, x)| \leq C(T-t)^{\frac{1}{2}}.
\]

**Proof. Step 1.** Fix \( i \) and let \( \hat{X}_{t_i} = x \in \mathcal{O} \). For \( j \geq i \), we define \( \hat{t}_j := t_j \land \hat{\tau} \) and \( \hat{j} := \frac{t_j}{h} \). Observe that by Lipschitz continuity of \( F \), for all \( j \) with \( j \geq i \), one can write
\[
v^h(\hat{t}_j, \hat{X}_{t_j}^i) = E_{\hat{t}_j} \left[ v^h(\hat{t}_{j+1}, \hat{X}_{t_j+1}^i) \left( 1 - \alpha_j + (a_j - b_j)h + (\sqrt{\alpha_j} \ln + \sqrt{\beta_j})^2 \right) + h \mathbb{1}_{\{\hat{\tau} > t_j\}} F^0_{t_j} \right],
\]
where \( F^0_{t_j} := F(t_j, \hat{X}_{t_j}^i, 0, 0, 0), 1 \geq \alpha_j \geq 0, a_j, b_j \) are \( \mathcal{F}_{t_j} \)-adapted and bounded random variables defined as in the proof of Lemma 3.11 and Lemma 3.16 in [2], \( a_i = 0 \) on \( \{ \hat{\tau} \leq t_j \} \), and \( N_j := \frac{W_{t_j+1} - W_{t_j}}{\sqrt{h}} \) which has a standard Gaussian distribution.

Combine the above formula for \( j \) from \( i \) to \( n-1 \), we see that
\[
v^h(t_i, x) = E \left[ g(\hat{\tau}, \hat{X}_{\hat{t}_i}^i) P_{i,n} \right] + h E \left[ \sum_{j=i}^{n-1} \mathbb{1}_{\{\hat{\tau} > t_j\}} \left( v^h(\hat{t}_{j+1}, \hat{X}_{t_j+1}^i) a_j + F^0_{t_j} \right) P_{t,j} \right],
\]
where \( P_{t,k} := \prod_{j=1}^{k-1} \left( 1 - \alpha_j - b_j h + (\sqrt{\alpha_j} \ln + \sqrt{\beta_j})^2 \right) \) and for all \( 1 \leq i < k \leq n \) and \( P_{i,i} = 1 \).

Since \( |F(\cdot, \cdot, 0, 0, 0)|_\infty \) and \( a_j \) are bounded, and by Lemma 4.11, one can conclude that
\[
E \left[ \sum_{j=i}^{n-1} \mathbb{1}_{\{\hat{\tau} > t_j\}} \left( v^h(\hat{t}_{j+1}, \hat{X}_{t_j+1}^i) a_j + F^0_{t_j} \right) P_{t,j} \right] \leq C E \left[ \sum_{j=i}^{n-1} \mathbb{1}_{\{\hat{\tau} > t_j\}} P_{t,j} \right].
\]

Observe that, since \( N_j \) is independent of \( \mathcal{F}_{t_j} \), \( \{ P_{t,j} \}_j \) is a martingale with respect to \( \{ \mathcal{F}_{t_j} \} \). This implies that
\[
|v^h(t_i, x) - g(t_i, x)| \leq \left| E \left[ (g(\hat{\tau}, \hat{X}_{\hat{t}_i}^i) - g(t_i, x)) P_{t,i} \right] \right| + C(T-t_i).
\]

**2.** Let \( \{ \rho_\varepsilon \} \) be a family of mollifiers and define \( \{ g_\varepsilon \}_\varepsilon \) by \( g_\varepsilon := g \ast \rho_\varepsilon \). Notice that we have
\[
|g_\varepsilon - g|_\infty \leq C \varepsilon, \| \partial_t g_\varepsilon \|_\infty \leq \varepsilon^{-1} \| \partial_t g \|_\infty, \| Dg_\varepsilon \|_\infty \leq \| Dg \|_\infty \text{ and } \| D^2 g_\varepsilon \|_\infty \leq C \varepsilon^{-1} \| Dg \|_\infty.
\]

Then, by Itô formula, we have
\[
g_\varepsilon(\hat{\tau}, \hat{X}_{\hat{t}_i}^i) - g_\varepsilon(t_i, x) = \int_{t_i}^{\hat{\tau}} \left( \partial_t g_\varepsilon + Dg_\varepsilon \hat{b} + \frac{1}{2} \text{Tr} \left[ \hat{a} D^2 g_\varepsilon \right] \right) (s, \hat{X}_{s}^i) ds + \int_{t_i}^{\hat{\tau}} (Dg_\varepsilon \hat{\sigma})(s, \hat{X}_{s}^i) dW_s,
\]
where we denoted \( \hat{b}(s) = b(t_j, \hat{X}^{t_i,x}_{t_j}) \) and \( \hat{\sigma}(s) = \sigma(t_j, \hat{X}^{t_i,x}_{t_j}) \) for \( t_j \leq s < t_{j+1} \) and \( \hat{a} = \hat{\sigma}^T \hat{\sigma} \). On the other hand, by using (4.2) and (4.3), one can write

\[
\left| \mathbb{E}\left[ \left( g(\hat{\tau}, \hat{X}^{t_i,x}_{\hat{\tau}}) - g(t_i, x) \right) P_{i,n} \right] \right| \leq \mathbb{E}\left[ \left| g(\hat{\tau}, \hat{X}^{t_i,x}_{\hat{\tau}}) - g_{\epsilon}(\hat{\tau}, \hat{X}^{t_i,x}_{\hat{\tau}}) P_{i,n} \right| \right] + \mathbb{E}\left[ \left| g(t_i, x) - g_{\epsilon}(t_i, x) P_{i,n} \right| \right] \\
\leq C|g| - g_{\epsilon}|_{\infty} + \mathbb{E}\left[ \left| g(\hat{\tau}, \hat{X}^{t_i,x}_{\hat{\tau}}) - g_{\epsilon}(t_i, x) \right| P_{i,n} \right] \leq C\varepsilon + \mathbb{E}\left[ \left| P_{i,n} \int_{t_i}^{\hat{\tau}} \left( \partial_1 g_{\epsilon} + Dg_{\epsilon} \hat{b} + \frac{1}{2} \text{Tr} \left[ \hat{a} D^2 g_{\epsilon} \right] \right) (s, \hat{X}^{t_i,x}_{s}) ds \right| \right] \\
+ \mathbb{E}\left[ P_{i,n} \int_{t_i}^{\hat{\tau}} (Dg_{\epsilon} \hat{\sigma})(s, \hat{X}^{t_i,x}_{s}) dW_s \right]. \tag{4.4}
\]

2.a. Since \( b \) and \( \sigma \) are bounded, by (4.2), we also have

\[
\left| Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) b(s, \hat{X}^{t_i,x}_{s}) + \frac{1}{2} \text{Tr} \left[ D^2 g_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) \hat{a}(s, \hat{X}^{t_i,x}_{s}) \right] \right| \leq C + C\varepsilon^{-1}. \tag{4.5}
\]

2.b. By martingale properties of \( \{P_{i,k}\} \), we have \( \mathbb{E}_{t_j+1}[P_{j+1,n}] = 1 \) which implies

\[
\mathbb{E}\left[ P_{i,n} \int_{t_i}^{\hat{\tau}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) \hat{\sigma}(s) dW_s \right] = \mathbb{E}\left[ P_{i,n} \sum_{j=1}^{n-1} \mathbbm{1}_{\{\hat{\tau} > t_j\}} \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) \hat{\sigma}(s) dW_s \right] \\
= \mathbb{E}\left[ \sum_{j=1}^{n-1} P_{i,j+1} \mathbbm{1}_{\{\hat{\tau} > t_j\}} \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) \hat{\sigma}(s) dW_s \mathbb{E}_{t_j+1}[P_{j+1,n}] \right] \\
= \mathbb{E}\left[ \sum_{j=1}^{n-1} P_{i,j} \hat{\sigma}(t_j) \mathbbm{1}_{\{\hat{\tau} > t_j\}} \mathbb{E}_{t_j} \left[ P_{j+1,n} \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) dW_s \right] \right].
\]

Notice that by isometry for Itô integral, one can write

\[
\mathbb{E}_{t_j} \left[ P_{j+1,n} \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) dW_s \right] = \mathbb{E}_{t_j} \left[ \left( \frac{\alpha_j}{h} (W_{t_{j+1}} - W_{t_j})^2 + 2b_j \sqrt{\alpha_j} (W_{t_{j+1}} - W_{t_j}) \right) \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) dW_s \right] \\
= \frac{\alpha_j}{h} \mathbb{E}_{t_j} \left[ \int_{t_j}^{t_{j+1}} 2W_s Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) ds \right] + 2b_j \sqrt{\alpha_j} \mathbb{E}_{t_j} \left[ \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) ds \right].
\]

Observe that since \( |Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) \hat{\sigma}(s)| \leq C\varepsilon \), we have

\[
\left| \mathbb{E}_{t_j} \left[ \int_{t_j}^{t_{j+1}} Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) ds \right] \right| \leq Ch. \tag{4.6}
\]

On the other hand, by Lemma 2.1 in \([\text{?}]\), one can conclude that

\[
\mathbb{E}_{t_j} \left[ \int_{t_j}^{t_{j+1}} W_s Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s}) ds \right] = \int_{t_j}^{t_{j+1}} \mathbb{E}_{t_j}[W_s Dg_{\epsilon}(s, \hat{X}^{t_i,x}_{s})] ds = \int_{t_j}^{t_{j+1}} s \mathbb{E}_{t_j}[D^2 g_{\epsilon}(s, \hat{X}^{t_i,x}_{s})] ds. \tag{4.7}
\]
Then, by (4.2), (4.7) implies that
\[
\mathbb{E} \left[ \int_{t_j}^{t_{j+1}} W_s Dg_\varepsilon(s, \hat{X}_s^{t_i,x}) ds \right] \leq C \varepsilon^{-1}.
\] (4.8)

Since \( \sigma \) is bounded, by (4.6), (4.8) and (4.1), one can conclude that
\[
\mathbb{E} \left[ \int_{t_i}^{t_i+\delta} Dg_\varepsilon(s, \hat{X}_s^{t_i,x}) \dot{\sigma}(s) dW_s \right] \leq C \varepsilon^{-1} \mathbb{E} \left[ \sum_{j=i}^{n-1} P_{i,j} \mathbf{1}_{\{ \tau > t_j \}} \right] \leq C \varepsilon^{-1} (T - t_i).
\]

\textbf{2.c.} Plugging (4.7), (4.6) and (4.8) into (1.3), we obtain
\[
\mathbb{E} \left[ \left( g_\varepsilon(\hat{\tau}, \hat{X}_{\hat{\tau}}^{t_i,x}) - g_\varepsilon(t_i,x) \right) P_{i,n} \right] \leq C(T - t_i)(1 + \varepsilon^{-1}),
\]
which by (1.4) provides
\[
|v^h(t_i, x) - g(t_i, x)| \leq C \varepsilon + C(T - t_i).
\]

The result follows from the choice \( \varepsilon = \sqrt{T - t_i} \) and \( \frac{1}{2} \)-Hölder continuity of \( g \) on \( t \). \( \square \)

\textbf{REFERENCES}


(Arash Fahim) \textsc{Department of Mathematics, University of Michigan, 530 Church Street, Ann Arbor, MI 48109, USA}

\textit{E-mail address:} fahimara@umich.edu