Section 10.2 Systems of Linear Equations in Three Variables

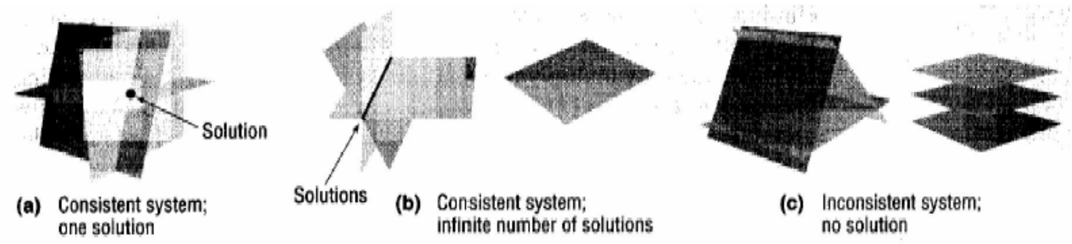
Geometric Meaning of Three Linear Equations in Three Variables

1. **Reminder of linear equation in three variables**

   A linear equation containing three variables, x, y and z, is an equation of the form Ax+By+Cz=D, where A, B, C and D are constants. The graph of such an equation is a PLANE in xyz-space.

2. **A system of three linear equations in three variables.**

   Thus, the system of three linear equations containing the variables, x, y and z, is a TRINARY of PLANEs in xyz-space. Each (x, y, z) trinary that satisfies the system of three equations must satisfy all three equations, i.e. the trinary (x, y, z) must be on all three planes.

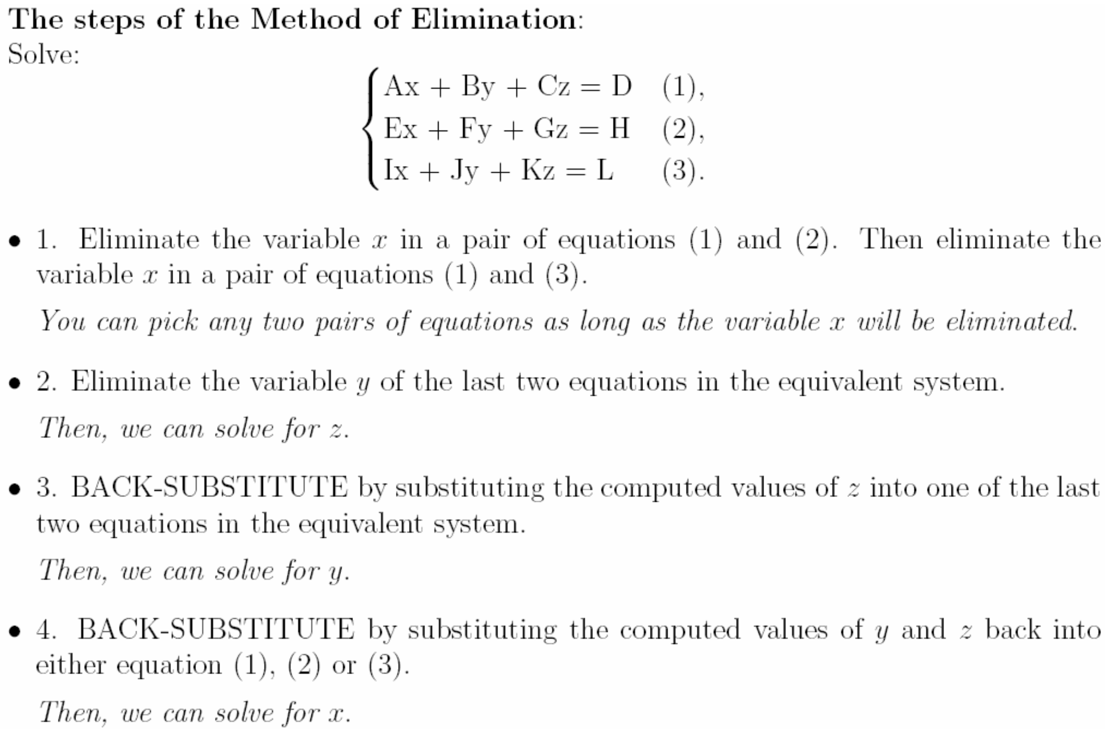


(a) Consistent system; one solution **(b)** Consistent system; infinite number of solutions **(c)** Inconsistent system; no solution

3. **Possible solutions of linear systems**
   - Exactly ONE solution (UNIQUE solution). The solution is exactly the point where the three planes which the three equations represent intersect.
   - INFINITELY MANY solutions. This is the second case, where the three line overlaps or their intersection forms a line.
   - NO solution. This is the third case, where the three planes have no point in common. There is no point that could be on all three plains.

   ATTN: IN NO CASE can a linear system has exactly two or three solutions.

Solving System of Equations by Elimination



The steps of the Method of Elimination:
Solve:

$$\begin{cases} \mathrm{Ax} + \mathrm{By} + \mathrm{Cz} = \mathrm{D} & (1), \\ \mathrm{Ex} + \mathrm{Fy} + \mathrm{Gz} = \mathrm{H} & (2), \\ \mathrm{Ix} + \mathrm{Jy} + \mathrm{Kz} = \mathrm{L} & (3). \end{cases}$$

- 1. Eliminate the variable x in a pair of equations (1) and (2). Then eliminate the variable x in a pair of equations (1) and (3).

  You can pick any two pairs of equations as long as the variable x will be eliminated.

- 2. Eliminate the variable y of the last two equations in the equivalent system.

  Then, we can solve for z.

- 3. BACK-SUBSTITUTE by substituting the computed values of z into one of the last two equations in the equivalent system.

  Then, we can solve for y.

- 4. BACK-SUBSTITUTE by substituting the computed values of y and z back into either equation (1), (2) or (3).

  Then, we can solve for x.

Note: You do not necessarily need to eliminate x first. The order of eliminating x, y or z and solving for x, y and z can be changed to your preference.

Example 1

[10.2.1PT]Select the type of solution for the following system

$$\begin{cases} x - y + z = -4 \\ 2x - 3y + 4z = -15 \\ 4x - 5y + 6z = -23 \end{cases}$$

○ Infinitely many solutions

○ No solution

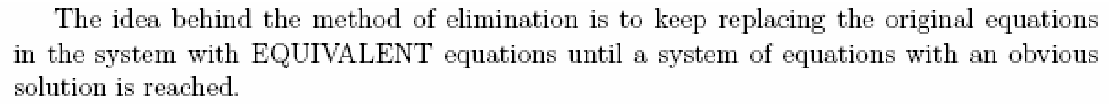
○ None of these

○ Unique solution

○ Exactly three solutions

Equivalent Systems Revisited

Recall: Two systems of linear equations are **equivalent** if the two systems have identical solutions.

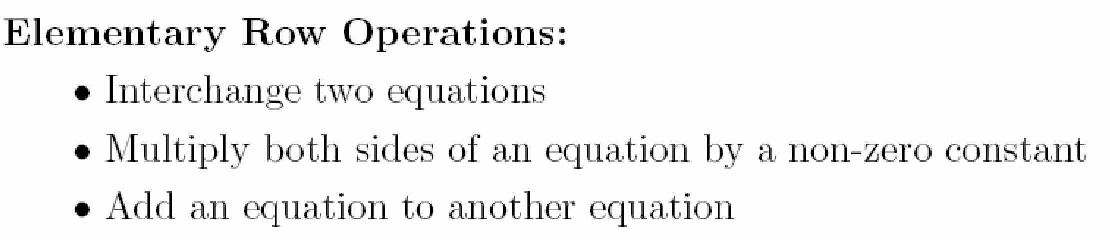


The idea behind the method of elimination is to keep replacing the original equations in the system with EQUIVALENT equations until a system of equations with an obvious solution is reached.

Let us have another look at the method of elimination we used in **example 1.**

- First step: $\begin{cases} R2 \leftarrow -2R1 + R2 \\ R3 \leftarrow -4R1 + R2 \end{cases}$
- Second step: $R3 \leftarrow -R2 + R3$

To generalize and sum up, the following Elementary Row Operations produce equivalent systems:

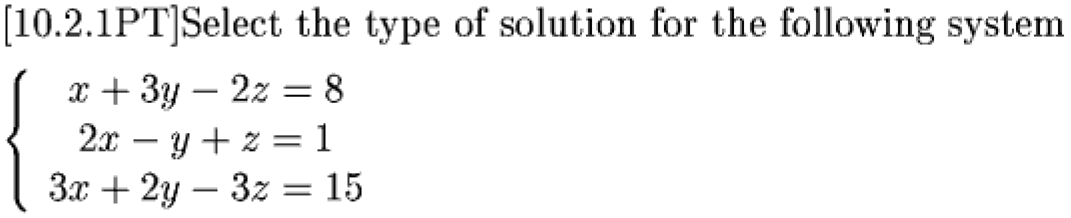


Elementary Row Operations:

- Interchange two equations
- Multiply both sides of an equation by a non-zero constant
- Add an equation to another equation

IMPORTANT Note: Operations like $Rm \leftarrow aRn + bRm$, where $a \neq 0$, is a composite of the 2nd and 3rd type of elementary row operations. Basically, we could regard "replacing a row with the sum of a multiple of another row and a multiple of itself" as a type of elementary row operation as well.

Exercise 2



[10.2.1PT]Select the type of solution for the following system

$$\begin{cases} x + 3y - 2z = 8 \\ 2x - y + z = 1 \\ 3x + 2y - 3z = 15 \end{cases}$$

○ Infinitely many solutions

○ None of these

○ Exactly three solutions

○ Unique solution

○ No solution