Section 10.3 Matrix

Linear System and Matrix

1. Coefficient matrix and augmented matrix

Example. The coefficient matrix, A, and the augmented matrix, B, for the following system are shown below.

$$2x - y + z = 3$$
$$x - y = 2$$
$$y + 2z = -3$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \text{ Coefficient matrix}$$

$$B = \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -3 \end{array}\right] \text{ Augmented matrix}$$

Definition of Matrix: A matrix is simply a rectangular array of numbers enclosed in parenthesis or brackets. (The order of the number matters.) Each number is call an entry/element of the matrix. A matrix with m rows and n columns is said to be of **order "m by n"**, often written "mxn". For example, the above coefficient matrix is of order 3x3 and the above augmented matrix is 3x4.

2. Equivalence between solving a linear system and simplifying[1] the matrix

Recall that we are essentially using elementary row operations to arrive at simpler equivalent systems.

Elementary Row Operations:

- Interchange two equations
- Multiply both sides of an equation by a non-zero constant
- Add an equation to another equation

We may also note that the x, y and z letters are redundant in writing down the linear system. All useful information can be fully represented by the augmented matrix. So there is a one-to-one correspondence between a linear system and its augmented matrix. Then applying elementary row operations to a linear system is equivalent to applying elementary row operations to a matrix.

Elementary Row Operations:

- Interchange two rows
- Replace a row with a non-zero multiple of the row
- Replace a row with the sum of the row and a multiple of another row

[1] By simplifying matrix, we mean that using elementary row operations to introduce more zeros into the matrix.

IMPORTANT Note: The operation like $Rm \leftarrow aRn + bRm$, where $a \neq 0$, is a composite of the 2nd and 3rd type of elementary row operations. Basically, we could regard "replacing a row with the sum of a multiple of another row and a non-zero multiple of itself" as a type of elementary row operation as well.

Exercise 1

[10.3.5aPT]Select the matrix obtained by applying the row operation $\mathbf{R_1} = -\mathbf{r_2} + \mathbf{r_1}$ to

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{bmatrix}$$

○ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & -2 & -15 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{bmatrix}$

EXERCISE 2

Pay close attention to the correspondence between equivalent linear systems and equivalent matrices. Pay attention to the elementary row operations and why we are applying them to the system and the matrix.

[10.3.4aPT]Choose the correct x, y, or z value for the solution of the system

$$\begin{cases} 3x + 5y - z = -7 \\ x + y + z = -1 \\ 2x + 11z = 7 \end{cases}$$

○ $x = -2$

○ $z = -2$

○ $z = -1$

○ $y = -1$

○ $y = 2$

Now, let us put aside the linear system and observe what happened to the augmented matrix.

- **We used a "small" number in the first column to annihilate other non-zero entries in the first column. This is exactly what the method of elimination does.**
- **Then we focus our elementary row operations on the submatrix in the down right part (excluding the first row and first column). This reflects that we focus on the simpler sub linear system.**
- **Now focus on the submatrix and repeat from the first step on until we arrive at a submatrix which is 1x1.**

Echelon Form and Reduced Echelon Form

The particular simplified form we just arrived at is almost the echelon form. Mathematicians now isolate this particularly interest form of matrix that people should pay attention to and abstracted the following definition for the echelon form.

1. **Definition of echelon form:** A matrix is in echelon form when
    - Each row containing a non-zero number has the number 1 appearing in the row's first non-zero column (the 1 is called **leading 1**);
    - The column numbers of the columns, containing the first non-zero entries in each of the rows, strictly increases from the first row to the last row

Note: This definition requires one-step further simplification of the equivalent matrix we arrive at in the last exercise. That is to require the leading non-zero element of each row to be 1 (achievable by the first kind of elementary row operation).

Remark: One should note the subtle but important point in the above definition. Although the idea of (augmented) matrix originates from the linear system, the definition above does not mention the related linear system at all. This small abstraction made by mathematicians looks trivial and insignificant so far. But this small step actually opened the door to the study of **Linear Algebra**. (There are many advanced courses covering this area.) Nowadays, Linear Algebra is extensively used in ALL fields of science and engineering. It is the basic language for scientific research. The study of Linear Algebra goes far beyond solving linear systems, although solving linear systems is still an important application of the theory of Linear Algebra.

Exercise 3

<u>Not</u> in Echelon Form

$$\begin{bmatrix}1&0&1&2\\0&0&1&1\\0&1&0&5\end{bmatrix}, \begin{bmatrix}1&0&1&2\\0&1&1&1\\0&0&0&5\end{bmatrix}, \begin{bmatrix}1&1&3\\0&0&2\end{bmatrix}, \begin{bmatrix}0&1&3\\1&0&2\end{bmatrix}$$

$$\begin{bmatrix}1&0&1&0\\0&0&0&0\\0&1&0&1\end{bmatrix}, \begin{bmatrix}-1&0&2&0\\0&1&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&-1\\0&-1&2\end{bmatrix}$$

In Echelon Form, but not Reduced Row Echelon Form:

$$\begin{bmatrix}1&1&3\\0&0&1\end{bmatrix}, \begin{bmatrix}1&2&3\\0&1&2\end{bmatrix}, \begin{bmatrix}1&0&1&2\\0&1&1&1\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&2&-1\\0&1&0&2\\0&0&0&1\end{bmatrix}$$

In Reduced Row Echelon Form:

$$\begin{bmatrix}1&0&3\\0&1&-1\end{bmatrix}, \begin{bmatrix}1&0&-1\\0&0&0\end{bmatrix}, \begin{bmatrix}1&2&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&2&0\\0&1&0&0\\0&0&0&1\end{bmatrix}, \begin{bmatrix}1&0&2&-1\\0&1&0&2\\0&0&0&0\end{bmatrix}$$

3. **Definition of REDUCED echelon form:** A matrix is in **reduced echelon form** when the matrix is in echelon form and all column entries above each of the row's first non-zero entries are zero.

Exercise 4

[10.3.3aPT]Select the matrix which is in reduced row echelon form

○ $\begin{bmatrix}1&-2&0&-5\\0&0&1&4\\0&0&0&1\end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

○ None of these

○ $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

○ $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Transforming A Matrix into Its Equivalent (Reduced) Echelon Form Echelon Form by ERO

Steps to produce echelon form

1. **Make the first entry of the first column to be 1 by interchanging two rows or multiplying the first row by a non-zero constant. If all the entries of the first column are already zero. Then pick the column to the right and do the same thing.**
2. **Use the 1 in the first column to annihilate the rest entries in the first column, i.e. make them zeros by ERO**
3. **Forget the column we were working on and the row contains the 1 we used. Then focus on the submatrix to the down right of the 1 we just used.**
4. **If the submatrix is not 1x1, regard the submatrix as the new matrix goto step 1 and repeat.**

Note: There could be some variations or small tricks. Check out the other notes in folder 10.3 for more details.

Exercise 5

[10.3.1aPT]Choose the echelon form of the following matrix

$$\begin{bmatrix} -4 & 1 & -2 \\ 2 & -4 & -4 \end{bmatrix}$$

○ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{10}{7} \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & \frac{10}{7} \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & \frac{6}{7} \\ 0 & 1 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Exercise 6

[10.3.1bPT]Choose the reduced row echelon form of the following matrix

$$\begin{bmatrix} 2 & -2 & 1 & 3 \\ 3 & 1 & -1 & 7 \\ 1 & -3 & 2 & 0 \end{bmatrix}$$

○ $\begin{bmatrix} 1 & 0 & 0 & \frac{7}{4} \\ 0 & 1 & 0 & -\frac{5}{4} \\ 0 & 0 & 1 & -3 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & \frac{9}{4} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & 1 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -17 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Steps to produce reduced echelon form

1. **Use ERO to obtain the echelon form first.**
2. **Use the leading 1's to annihilate the non-zero entries above them. We need to start with the lowest leading 1.**

Note: There could be some variations or small tricks. Check out the other notes in folder 10.3 for more details.

Exercise 7

Complete **Exercise 5.**

Exercise 8

[10.3.1bPT]Choose the reduced row echelon form of the following matrix

$$\begin{bmatrix} 2 & -2 & -4 & -2 \\ 3 & -3 & -6 & -3 \\ -2 & 3 & 1 & 7 \end{bmatrix}$$

○ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 0 & -\frac{13}{3} \\ 0 & 1 & 1 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & -1 & -2 & -1 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & -5 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 & 0 & 1 & -\frac{4}{5} \\ 0 & 1 & 0 & \frac{13}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$