Section 10.3 Determinants

Determinants

Remark: Determinants are scalars related to square matrices. They kind of serve the role as the “abosolute” value of matrices, although determinants could be a negative number.

Determinants are important mathematical tools that can frequently be used to analyz linear systems. There are various methods for finding determinants. Some of the simp only apply to $2x2$ and $3x3$ matrices.

Determinants of $2x2$ matrices

If $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2 by 2 matrix, then the **determinant** of D is

- $det(D) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example. Find the determinant of $D = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix}$

Solution.

$$\begin{vmatrix} 1 & -1 \\ 2 & -3 \end{vmatrix} = (1)(-3) - (-1)(2) = -1$$

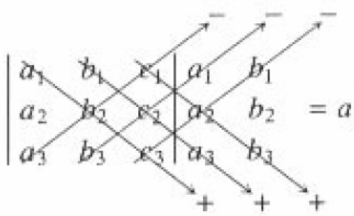
Determinants of $3x3$ matrices

If $D = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a $3x3$ matrix, then the determinant of D is

- $det(D) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32})$$

The following is a good device for remembering how to evaluate $3x3$ determinants.



$$= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

Exercise 1

Example. Find the determinant of

$$D = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix}$$

Solution.

$$\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ -1 & -2 & -1 \end{vmatrix}$$
$$= (1)(1)(-1) + (-1)(1)(-1) + (0)(2)(-2) - ((0)(1)(-1) + (-1)(2)(-1) + (1)(1)(-2))$$

$$= -1 + 1 + 0 - (0 + 2 - 2) = 0$$

Important note. The methods shown above for $2x2$ and $3x3$ determinants does NOT apply to $4x4$ or higher-order determinants.

Exercise 2

$$[10.4.1\text{aPT}] \begin{vmatrix} -1 & 2 & 3 \\ -3 & 4 & 5 \\ x & -y & 2z \end{vmatrix} =$$

○ $-x + 2y + 2z$

○ $2x - 4y - 4z$

○ $x - 2y - 2z$

○ $-2x + 4y + 4z$

Cramer’s Rule

Remark: Cramer’s rule is a formally elegant way to solve a linear system. But, in real life, people do not use it to solve linear system because it is computationally costly.

Cramer’s Rule

Cramer’s rule is a method that can be used for solving linear systems when the number of unknowns is the same as the number of equations and the determinant of the system’s coefficient matrix is not equal to zero.

Cramer's Rule for the case of two unknowns:

Consider the following system.

$$ax + by = s$$
$$cx + dy = t$$

If $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$ in the above system, then the above system has a unique solution given b

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Example. Solve the following system using Cramer's rule.

$$3x - 2y = 1$$
$$x + 2y = 2$$

Solution. By Cramer's Rule

$$x = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{(1)(2)-(2)(-2)}{3(2)-(-2)(1)} = \frac{6}{8} = \frac{3}{4}$$

$$y = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{(3)(2)-(1)(1)}{3(2)-(-2)(1)} = \frac{5}{8}$$

Cramer's Rule for the case of three unknowns:

Consider the following system.

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

If, in the above system,

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then the above system has a unique solution given by

$$x = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{D} \qquad y = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{D} \qquad z = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{D}$$

Exercise 3

Example. Use Cramer's rule to solve the following system.

$$x - y + z = 0$$
$$2x + y - z = 1$$
$$x - y + 2z = -2$$

Solution. Let

$$D = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 3$$

Then, by Cramer's rule,

$$x = \frac{\begin{vmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ -2 & -1 & 2 \end{vmatrix}}{D} = \frac{1}{3}$$

$$y = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix}}{D} = -\frac{5}{3}$$

$$z = \frac{\begin{vmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & -2 \end{vmatrix}}{D} = \frac{-6}{3} = -2$$

Exercise 4

[10.4.2aPT]Select the solution given by Cramer's rule for the following system, where

$D = \begin{vmatrix} 1 & 4 & -2 \\ 3 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$ $A = \begin{vmatrix} 0 & 4 & -2 \\ 4 & -2 & 3 \\ -1 & 1 & -3 \end{vmatrix}$ $B = \begin{vmatrix} 1 & 0 & -2 \\ 3 & 4 & 3 \\ 2 & -1 & -3 \end{vmatrix}$ $C = \begin{vmatrix} 1 & 4 & 0 \\ 3 & -2 & 4 \\ 2 & 1 & -1 \end{vmatrix}$

$$\begin{cases} x + 4y - 2z = 0 \\ 3x - 2y + 3z = 4 \\ 2x + y - 3z = -1 \end{cases}$$

- ○ $y = B/D$
- ○ $y = D/B$
- ○ $y = C/D$
- ○ $y = D/C$
- ○ None of these

Determinants of Larger Square Matrices

In order to generalize the determinants to higher order matrices, we need to use another definition for the determinants. Interested readers may google or wiki for detailed definition and illustration.