Section 10.5 Matrix Algebra

Introduction

1. Recall:

A **matrix** is a rectangular array of numbers. A matrix that has m **rows** and n **columns** is said to be an $m \times n$ matrix, and the $m \times n$ matrix is said to be of **order** $m \times n$. The element a_{ij} is in row i and column j of the matrix (a_{ij}).

$$(a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Examples of matrices.

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -2 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 2 \\ 3 & 4 & 17 \end{pmatrix}$$

2. Understanding of Matrix

The matrix could be regarded as generalization to numbers. We are familiar with real numbers and complex numbers. Matrices are arrays of such numbers. We have equipped numbers with equality and such operations as addition, subtraction, multiplication and division. Similarly, we are going to equip matrices with **similar operations (But multiplication and division are more involved. BE CAREFUL!)**

The reason that multiplication and division for matrices are more restrictive than those for numbers is that the matrix algebra and the number algebra belong to two different kind of mathematical algebra structures. Matrices form an algebraic structure called **ring** while the numbers form an algebraic structure called **field**. Interested readers may google or wikipedia these two important and fundamental types of algebra.

3. Matrix Equality

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are **equal** iff $a_{ij} = b_{ij}$ for all i, j. Note: Two matrices are equal iff corresponding entries are identical.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Addition, Subtraction and Scalar Multiplication

Only matrices of the same order can be added and subtracted. Matrices are added and subtracted term by term, i.e., if $A = (a_{ij})$ and $B = (b_{ij})$, then $A+B = (a_{ij}+b_{ij})$ and $A-B = (a_{ij}-b_{ij})$.

Example. If $A = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ -4 & 7 \end{pmatrix}$, then

$$A + B = \begin{pmatrix} 3 & 1 \\ -6 & 10 \end{pmatrix}$$

$$A - B = \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}$$

Let α be a scalar and let $A = (a_{ij})$ be a matrix. Then **scalar multiplication** is defined by $\alpha A = (\alpha a_{ij})$.

Example. Let $\alpha = 2$ and $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$. Then

$$\alpha A = 2\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -4 & 6 \end{pmatrix}$$

Some Properties of Matrix Addition

$A + B = B + A$

$A + (B + C) = (A + B) + C$

$(k + h)A = kA + hA$

$k(A + B) = kA + kB$

Multiplication of Matrix

1. Multiplication of a row vector and a column vector.

The $1 \times n$ matrix $R = [r_1, r_2, \cdots, r_n]$ is called a **row vector**. The $m \times 1$ matrix

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix}$$

is called a **column vector**.

The **product** RC, of a $1 \times n$ row vector and an $m \times 1$ column vector, is defined only when $m = n$. If $m = n$, then

$$RC = \begin{pmatrix} r_1 & r_2 & \cdots & r_m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = (r_1c_1 + r_2c_2 + \cdots + r_mc_m)$$

Example. If $R = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, then

$$RC = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = (2(1) + 0(-1) + 2(1)) = (4)$$

2. Multiplication of two matrices

Let $A = (a_{ij})$ be $m \times n$ and let $B = (b_{ij})$ be $n \times r$; then $AB = (c_{ij})$ is defined and c_{ij}, the element of $C = AB$ in row i and column j, is given by

$$c_{ij} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix}$. Then A is 3×2 and B is 2×2. The product AB is defined, and AB is 3×2. If

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{pmatrix}, \text{ then}$$

$$c_{11} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -8$$

$$c_{12} = \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$$

$$c_{21} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -1$$

$$c_{22} = \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2$$

$$c_{31} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} -2 \\ -3 \end{pmatrix} = -18$$

$$c_{32} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3$$

Hence, $AB = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} -8 & 1 \\ -1 & 2 \\ -18 & 3 \end{pmatrix}$

Remark: For multiplication of matrices, one must make sure that the column number of the left matrix is the same as the row number of the right matrix. Otherwise, we CANNOT multiply them together.

Some properties:

- $AB \neq BA$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$

where A, B, C are matrices with the appropriate orders.

Exercise 1

[10.5.1aPT]Find $BC - 2B$ if

$$B = \begin{pmatrix} -3 & 1 \\ 2 & -4 \\ 1 & -2 \end{pmatrix} \quad C = \begin{pmatrix} -6 & 3 \\ 9 & -3 \end{pmatrix}$$

○ $\begin{pmatrix} 21 & -10 \\ -44 & 10 \\ -22 & 5 \end{pmatrix}$

○ $\begin{pmatrix} 33 & -14 \\ -52 & 26 \\ -26 & 13 \end{pmatrix}$

○ can not be done

○ $\begin{pmatrix} 21 & -14 \\ -52 & 10 \\ -22 & 5 \end{pmatrix}$

Exercise 2

[10.5.1bPT]Find the matrix product ABC if

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \; B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 0 & -1 \end{pmatrix} \; C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

○ $\begin{pmatrix} -4 & -2 \\ -5 & 5 \end{pmatrix}$

○ $\begin{pmatrix} 1 & -2 \\ -8 & 1 \end{pmatrix}$

○ $\begin{pmatrix} -2 & -6 \\ -3 & 1 \end{pmatrix}$

○ $\begin{pmatrix} -3 & -4 \\ 0 & 5 \end{pmatrix}$

The Inverse of Matrix

Let's again heuristically set up an analogy between scalar numbers and matrices.

- $1 \leftrightarrow I$
- $a^{-1} \leftrightarrow A^{-1}$
- $a^{-1}a = aa^{-1} = 1 \leftrightarrow A^{-1}A = AA^{-1} = I$
- $|a| = 0 \Leftrightarrow a^{-1} \sim does \sim not \sim exist \leftrightarrow |A| = 0 \Leftrightarrow A^{-1} \sim does \sim not \sim exist$

Here are the definitions for the new notations we used above:

- **Identity matrix** is a square matrix whose *diagonal entries* are all identically 1 and the rest of the entries are identically 0.
- A^{-1} denotes a matrix, called the inverse of matrix A, such that $A^{-1}A = AA^{-1} = I$
- $|A| = 0 \Leftrightarrow A$ is not invertible. In this case, A is called **singular**. Otherwise, A is called **non-singular**.

Procedure for finding a matrix inverse

- Start with the matrix $(A|I)$.
- Use row operations to transform the above matrix to $(I|B)$, if possible.
- Then $A^{-1} = B$.
- If we obtain a row of zeros to the left of the vertical line, the inverse of A does not exist.

Remark: We are actually using the skills we learned to transform a matrix into its reduced echelon form.

Exercise 3

[10.5.2c1PT]Select the entry in the first row and second column of the inverse matrix for

$$\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

○ 3
○ 1
○ 2
○ None of these
○ -1

Exercise 4

[10.5.2cPT]Select the entry in the third row and second column of the inverse matrix for

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$$

○ -6
○ -1
○ 2
○ 3

Exercise 5 (NO NEED TO FIND THE EXACT INVERSE)

[10.5.2aPT]Select the third row of the inverse matrix for

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 3 & 1 \\ 2 & -1 & 2 \end{pmatrix}$$

○ (6 -3 2)

○ (-6 -5 3)

○ (-6 5 -3)

○ (6 -5 3)

Exercise 6 (NO NEED TO FIND THE EXACT INVERSE)

[10.5.2bPT]Select the third column of the inverse matrix for

$$\begin{pmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{pmatrix}$$

○ $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

○ $\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}$

○ $\begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}$

○ $\begin{pmatrix} -3 \\ 6 \\ 5 \end{pmatrix}$

The Inverse of Matrix and Linear System

- Write the linear system in this form $AV = R$, where A is the $n \times n$ coefficient matrix, V is the column vector of n variables, and R is the column vector of the right-hand side of the system.
- Find the matrix inverse A^{-1} if it exists.
- Then $A^{-1}AV = A^{-1}R \Leftrightarrow V = A^{-1}R$.

Exercise 7

[10.5.2dPT]Find y in the solution of the system

$$\begin{cases} a_1x + b_1y = -\frac{1}{2} \\ a_2x + b_2y = 1 \end{cases}$$

if the inverse of the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is $\begin{bmatrix} -2 & -\frac{1}{3} \\ 4 & \frac{1}{2} \end{bmatrix}$

○ $\frac{2}{3}$

○ $-\frac{2}{3}$

○ $-\frac{3}{2}$

○ 5

Exercise 8

[10.5.2ePT]Find z in the solution of the system

$$\begin{cases} a_1x + b_1y + c_1z = \frac{1}{4} \\ a_2x + b_2y + c_2z = -2 \\ a_3x + b_3y + c_3z = -1 \end{cases}$$

if the inverse of the coefficient matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is $\begin{bmatrix} -2 & 1 & -\frac{1}{2} \\ -1 & -1 & \frac{1}{4} \\ 2 & -\frac{1}{4} & -1 \end{bmatrix}$

○ $\frac{5}{2}$

○ $\frac{3}{2}$

○ $\frac{3}{8}$

○ $-\frac{1}{2}$

- ○ 2
- ○ -2