Section 11.4 Mathematical Induction

Statement Sequence

Definition: A **statement sequence** is a sequence $\{S_n\}$, whose terms are statements (or formula) parameterized by the index n.

Example of statement sequence

$$\{1=1, 1+2=\frac{(1+2)2}{2}, 1+2+3=\frac{(1+3)3}{2}, 1+2+3+4=\frac{(1+4)4}{2}, \cdots, \sum_{k=1}^{n} k=\frac{(1+k)k}{2}, \cdots\}$$

Here, $S_n : \sum_{k=1}^{n} k = \frac{(1+k)k}{2}$, is the statement of a general index n.

Note: In the above example, the n-th statement is essentially claiming a formula for the finite sum of a particular arithmetic sequence $\{1, 2, 3, 4, \cdots\}$, whose first tem is 1 and whose common difference is 1.

How do we prove that the all of the statements in the sequence are true?

1. Verify the validity of each statement. It is usually easy to verify the first a couple of statements. But there are infinitely many statements…
2. To prove the validity of the n-th statement DIRECTLY. But in most cases, this is prohibitively hard.

Principles of Mathematical Induction

We can prove that all of the statements in a statement sequence are true if we can

- Show the first statement, S_1, is true. (**Basic Step**)
- Show that IF S_k is true for a particular k, THEN S_{k+1} (the next statement) must also be true. That is show $S_k \Rightarrow S_{k+1}$. (**Inductive Step**)

(**The important issue on solving problems of this section is to identify the statement with any given index n.**)

Exercise 1

[11.4.1aPT]Find a_2 and a_3 such that $1 + a_2 + a_3 + \cdots + a_n = \frac{n(3n-1)}{2}$ for all n.

○ $a_2 = 4, a_3 = 7$

○ $a_2 = 5, a_3 = 9$

○ None of these

○ $a_2 = 5, a_3 = 12$

Exercise 2

[11.4.2aPT]To prove by induction that $9+7+5+\cdots+(11-2n) = 10n-n^2$ is true for all positive integers n, we assume $9+7+5+\cdots+(11-2k) = 10k-k^2$ is true for some positive integer k, and show that $9+7+5+\cdots+(11-2k)+A = 10(k+1)-(k+1)^2$ where A is

○ $13-2k$
○ $9-2k$
○ $10-2k$
○ $12-2k$
○ None of these

Exercise 3

[11.4.2bPT]To prove by induction that $10+7+4+\cdots+(13-3n) = \frac{1}{2}(23n-3n^2)$ is true for all positive integers n, we assume $10+7+4+\cdots+(13-3k) = \frac{1}{2}(23k-3k^2)$ is true for some positive integer k, and show that $10+7+4+\cdots+(13-3k)+(13-3(k+1)) = A$ where A is

○ $\frac{1}{2}((23k+1)-(3k^2+1))$

○ $\frac{1}{2}(23(k+1)-3(k+1)^2)$

○ $\frac{1}{2}(23(k+1)-(3k+1)^2)$

○ $\frac{1}{2}(23(k+1)-3(k+1)^2)+1$

○ $\frac{1}{2}(23k-3k^2)+1$

Exercise 4

[11.4.3aPT]To prove by induction that n^2-5n-2 is divisible by 2 is true for all positive integers n, we assume k^2-5k-2 is divisible by 2 is true for some positive integer k, and we show that A is divisible by 2, where A is

○ $k^2-5k-2+1$

○ $(k^2+1)-(5k+1)-2$

○ None of these

○ $(k+1)^2-5(k+1)-2$

○ $(k+1)^2-5(k+1)-2+1$

Exercise 5

[11.4.3bPT]To prove by induction that n^2-7n-4 is divisible by 2 is true for all positive integers n, we assume k^2-7k-4 is divisible by 2 is true for some positive integer k and we show that $k^2-7k-4+A$ is divisible by 2, where A is

○ None of these

○ $2(k-2)$

○ $2(k-1)$

○ $2(k+3)$

○ $2(k-3)$