# Section 3.7 Rational Functions

# (Must understand 3.3, Come back to real numbers)

## Rational Function

1. $f(x) = \frac{p(x)}{q(x)}$, where p(x) and q(x) are two polynomials.

2. **Domain:** The domain of f(x) defined above is the set of all (real) x such that $q(x) \neq 0$.

   So if the q(x) does not have a zero, then the domain is all real numbers.

**Exercise 1**

[3.7.1aPT]Choose the domain of the rational function $f(x) = -\frac{3x}{x^2-9}$

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

$(-\infty, -3) \cup (-3, 0) \cup (0, 3) \cup (3, \infty)$

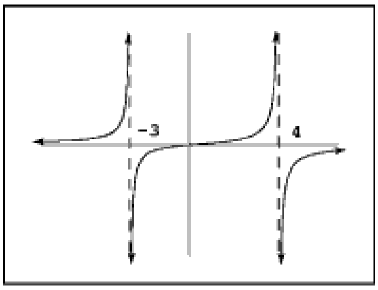
$(-\infty, 0) \cup (0, \infty)$

$(-\infty, \infty)$

$(-\infty, -3) \cup (-3, 1) \cup (1, 3) \cup (3, \infty)$

**Exercise 2**

[3.7.1bPT]Choose the domain of the rational function shown below




$(-\infty, -3) \cup (4, \infty)$

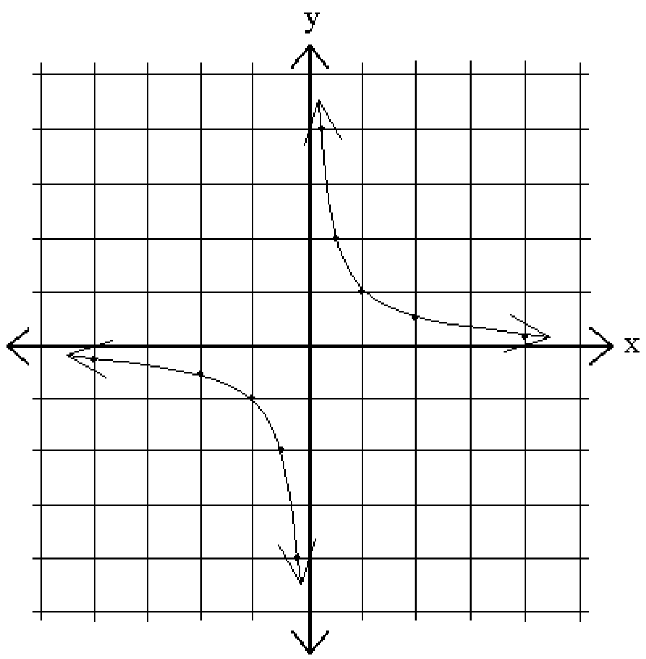
$(-\infty, -3) \cup (-3, 0) \cup (0, 4) \cup (4, \infty)$

$(-3, 4)$

$(-\infty, \infty)$

$(-\infty, -3) \cup (-3, 4) \cup (4, \infty)$

**3. Recall Inverse Variation:** $f(x) = \frac{1}{x}$




Note that the lines never cross the axes -- they get closer and closer to x = 0 and y = 0, but x and y never equal zero. (See the supplementary material for more details.)

## Vertical Asymptote and Behavior near the Zeros of the Denominator

The function blows up, shooting vertically up or down, near the zeros of the denominator, just as inverse variation function does. For example, if $f(x) = \frac{p(x)}{q(x)}$ and r is a zero of q(x), then the graph of the function would approach (upward or downward) the vertical line x=r, but never touches it. Such vertical line is called **vertical asymptote.**

More formally, the line of x=r is a **vertical asymptote** of the graph of f(x) if $|f(x)| \to \infty$ as $x \to r$.

**Remark:** A rational function could have many vertical asymptotes. One zero of q(x) correspond to one vertical asymptote.

**Exercise 3**

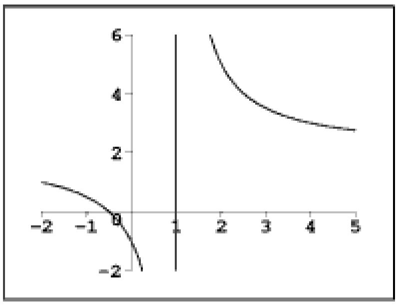
[3.7.2aPT]Select the choice having ALL the vertical asymptotes of the function $f(x) = \frac{x^2+1}{(x-2)^2}$

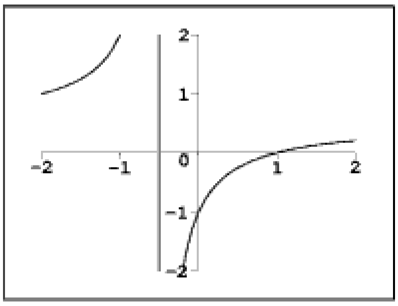
$(2, 0)$

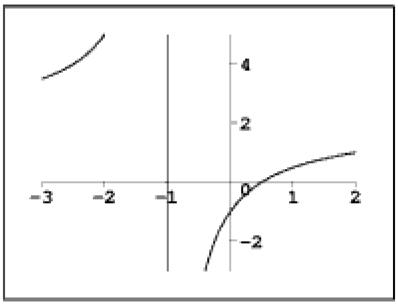
$y = 2x$
$x = 2$,$x = -2$
$x = 2$
$x = 2,\ x = 1,\ x = -1$
$y = 2$

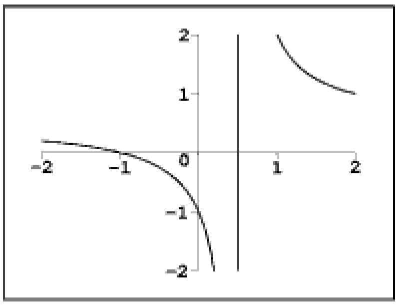
**Exercise 4**

[3.7.4aPT]Select the graph of $y = \frac{2x-1}{x+1}$.



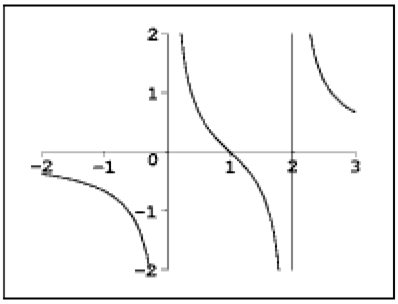






**Exercise 5**

[3.7.4bPT]Select the equation of the following graph.




$y = \frac{1-x}{x(x-2)}$

$y = -\frac{x+1}{x(x+2)}$

$y = \frac{x-1}{x(x-2)}$

$y = \frac{x+1}{x(x+2)}$

## Horizontal Asymptote

Let $f(x) = \dfrac{p(x)}{q(x)}$, the degree of p(x) is m, the degree of q(x) is n. Then when $m \le n$, the graph of f(x) will have a horizontal asymptote.

- If m<n (i.e. f(x) is proper), then y=0 is the horizontal asymptote.
- If n=m, then $y = \dfrac{a_m}{b_n}$ is the horizontal asymptote, where $a_m$ is the leading coefficient of p(x) (including the sign), $b_n$ is the leading coefficient of q(x) (including the sign).
- Otherwise, NO horizontal asymptotes.

**Remark:** A rational function has at most one horizontal asymptote.

**Exercise 6**

[3.7.2bPT]Select the choice having ALL the horizontal asymptotes of $f(x)=\frac{x(x+2)}{1-x^2}$

$x=-1$

No horizontal asymptotes

$y=-1$, $y=1$

$y=-1$

$y=-1$, $y=0$, $y=-2$

$y=-1$, $y=0$

## Oblique Asymptote

Let $f(x)=\frac{p(x)}{q(x)}$, the degree of p(x) is m, the degree of q(x) is n. Then when $m=n+1$, then there is an oblique asymptote, which is the QUOTIENT computed by long division. Otherwise, NO oblique asymptote.

**Exercise 7**

[3.7.3aPT]Find the oblique asymptote of $f(x)=\frac{2x^2}{x-1}$

$y=2x+2$

$y=3x-1$

$y=2x-2$

No oblique asymptote

***Extra Credit***

1. Use the behavior of inverse variation function near x=0 to explain the behavior of a general rational function near the zeros of its denominator.
2. Use the theory of ends behavior of polynomials to explain to explain the following facts

- When m<n, y=0 is the horizontal asymptote;
- When m=n, $y=\frac{a_m}{b_n}$ is the horizontal asymptote.
- When m=n+1, the quotient of $\frac{p(x)}{q(x)}$ is the oblique asymptote. (Hint: Put $\frac{p(x)}{q(x)}$ into the form $\frac{p(x)}{q(x)}=quotient(x)+\frac{r(x)}{q(x)}$. What's the degree of quotient(x)? What the ends behavior of $\frac{r(x)}{q(x)}$?)
- Otherwise, no horizontal asymptote nor oblique asymptote.