

Section 4.1 One-to-One Functions

(Section 2.1 needed)

One-to-One Functions

1. **Function:** A function $f(x)$ can be thought of as a machine, whose input is a number x and output is a UNIQUE number $y=f(x)$, i.e., for each x , there is one and only one y such that $y=f(x)$. The number x is called a **sausage of $f(x)$** , and $y=f(x)$ is called the **image of that sausage**.

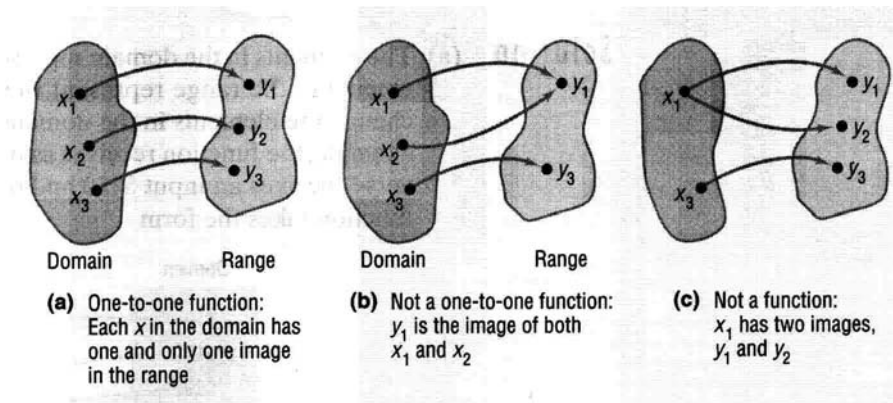
Remark: We use vertical line test to check if a graph is a graph of a function. (See 2.1.)

2. **One-to-One Function:** A function is one-to-one if for two DIFFERENT inputs, x_1 and x_2 , the outputs $f(x_1)$ and $f(x_2)$ are DIFFERENT.

Mathematically, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, i.e. if $f(x_1) = f(x_2)$ then $x_1 = x_2$. (contrapositive of the conditional)

Remark: We use horizontal line test to check if a function is one-to-one.

Horizontal line test: A function f is one-to-one if and only if every horizontal line intersects the graph of f in AT MOST ONE point.



Example 1

[4.1.1aMSPT] Select the formulas of ALL one-to-one functions

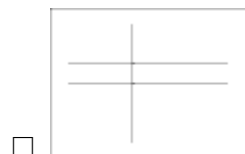
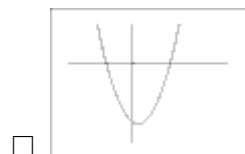
- $f(x) = -4$
- $f(x) = -3x^2 - 2x + 1$
- $f(x) = -2x + 3$

$f(x) = -\frac{1}{5}x^3 + \frac{1}{3}$

$f(x) = \frac{-x+3}{x-2}$

Exercise 2

[4.1.1bMSPT] Select the graphs of ALL one-to-one functions



Inverse Function

1. **Definition:** The inverse function of a ONE-TO-ONE function, $f(x)$, is denoted by $f^{-1}(x)$ (**not reciprocal**). If (a,b) is a point on the graph of $y = f(x)$, then (b,a) is a point on the graph of $y = f^{-1}(x)$.

Remark: The x in $f(x)$ and the x in $f^{-1}(x)$ are **NOT** the same x . They just denote the independent variable of the function.

Mathematically, given a function f , the inverse function f^{-1} satisfies

$f^{-1}(f(x)) = x$ (in this identity the x 's refer to the same number) for all x in the

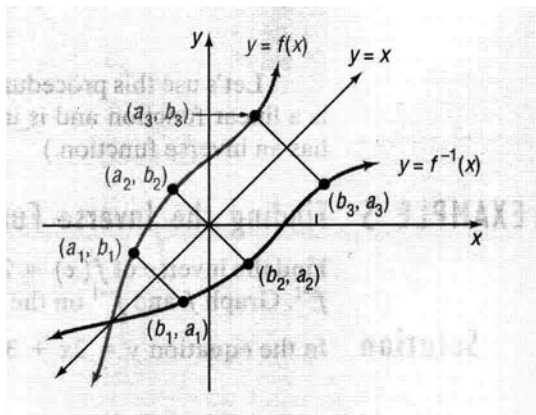
domain of f . In other words, the inverse function f^{-1} undoes whatever $f(x)$ does to recover the input x of the machine f , i.e. f^{-1} reverses the one-to-one mapping created by f .

2. Properties:

- Range of f = Domain of f^{-1}
- Domain of f = Range of f^{-1}
- The graph of f and its inverse f^{-1} are symmetric about the line $y=x$ (a result of the (a,b) and (b,a) correspondence), i.e. we may reflect the graph of f about the line $y=x$ to find the graph of f^{-1} .

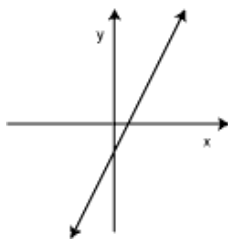
Finding the Inverse of a Function

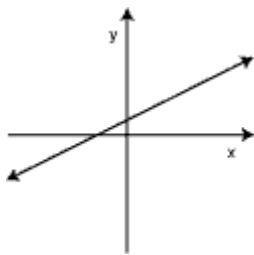
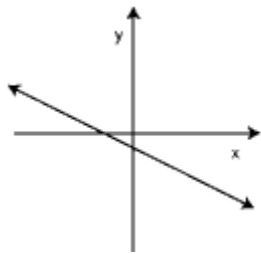
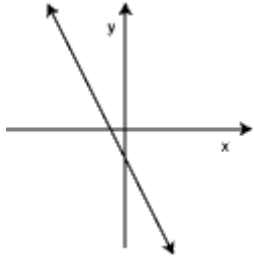
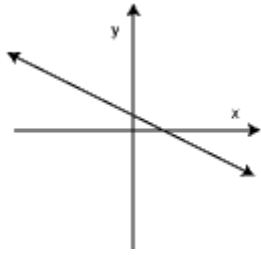
1. Graphically:



Exercise 3

[4.1.3aPT] Select the graph of the inverse of the function shown below





This function has no inverse

2. Algebraically:

- Is f an one-to-one function? If not, NO INVERSE. Otherwise, go to next step.
- Determine the domain and range of f .
- Solve the expression $y = f(x)$ for x to obtain $x = f^{-1}(y)$ (if possible)
- Interchange the variables x and y to obtain the expression $y = f^{-1}(x)$.
- **The domain of the inverse $y = f^{-1}(x)$ is the range of $y = f(x)$. Don't forget!**

Exercise 4

[4.1.2bPT] If $f(x) = a(x - 3)^2 + 2$ on $(3, \infty)$ and $a > 0$, then the inverse function is

- $f^{-1}(x) = -\sqrt{\frac{x-2}{a}} + 3$ on $(3, \infty)$
- $f^{-1}(x) = -\sqrt{\frac{x-2}{a}} + 3$ on $(2, \infty)$
- $f^{-1}(x) = \frac{1}{a(x-3)^2+2}$ on $(3, \infty)$
- $f^{-1}(x) = \sqrt{\frac{x-2}{a}} + 3$ on $(3, \infty)$
- $f^{-1}(x) = \sqrt{\frac{x-2}{a}} + 3$ on $(2, \infty)$

Exercise 5

[4.1.2cPT] If $f(x) = 4x + 2$ on $[-2, 2]$, then the inverse function is

- $f^{-1}(x) = \frac{1}{4x+2}$ on $[-2, 2]$
- $f^{-1}(x) = \frac{1}{4x+2}$ on $[-6, 10]$
- None of these
- $f^{-1}(x) = \frac{1}{4}x - \frac{1}{2}$ on $[-2, 2]$
- $f^{-1}(x) = \frac{1}{4}x - \frac{1}{2}$ on $[-6, 10]$