

**Foundations of Computational Math I Exam 1**  
**Take-home Exam**  
**Open Notes, Textbook, Homework Solutions Only**  
**Calculators Allowed**  
**No collaborations with anyone**  
**Due beginning of Class Wednesday, October 26, 2011**

Question	Points Possible	Points Awarded
1. Basics	25	
2. Linear operators	25	
3. Floating point	25	
4. Factorization	25	
5. Orthogonal Factorization	25	
Total Points	125	

**Name:** \_\_\_\_\_

**Alias:** \_\_\_\_\_

to be used when posting anonymous grade list.

# Problem 1

(25 points)

## 1.a

(10 points)

Suppose  $A \in \mathbb{R}^{m \times n}$  and consider the matrix 2-norm

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$$

Show that  $\|A\|_2 \geq \|A_1\|_2$  where

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

$m = m_1 + m_2$ ,  $A_1 \in \mathbb{R}^{m_1 \times n}$ , and  $A_2 \in \mathbb{R}^{m_2 \times n}$ .

## 1.b

**(15 points)**

Let  $\mathcal{S}_1 \subset \mathbb{R}^n$  and  $\mathcal{S}_2 \subset \mathbb{R}^n$  be two subspaces of  $\mathbb{R}^n$ .

- (i) **(5 points)** – Suppose  $x_1 \in \mathcal{S}_1$ ,  $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$ ,  $x_2 \in \mathcal{S}_2$ , and  $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$ . Show that  $x_1$  and  $x_2$  are linearly independent.

- (ii) **(10 points)** – Suppose  $x_1 \in \mathcal{S}_1$ ,  $x_1 \notin \mathcal{S}_1 \cap \mathcal{S}_2$ .  $x_2 \in \mathcal{S}_2$ , and  $x_2 \notin \mathcal{S}_1 \cap \mathcal{S}_2$ . Also, suppose that  $x_3 \in \mathcal{S}_1 \cap \mathcal{S}_2$  and  $x_3 \neq 0$ , i.e., the intersection is not empty. Show that  $x_1$ ,  $x_2$  and  $x_3$  are linearly independent. (Note the result of the previous part of the problem may be useful.)

## Problem 2

(25 points)

### 2.a

(15 points)

Recall that  $\mathcal{P}_n$ , the set of polynomials of degree less than or equal to  $n$ , and the operation of polynomial addition is equivalent to the vector space  $\mathbb{C}^{n+1}$ .

- (i) **(5 points)** Show that the mapping from a polynomial  $p(\tau) \in \mathcal{P}_n$  to its derivative with respect to  $\tau$ ,  $p'(\tau) \in \mathcal{P}_n$  can be expressed as an  $n+1 \times n+1$  matrix applied to a vector  $v$ , i.e.,  $v' = Dv$ , where the vector  $v \in \mathbb{C}^{n+1}$  represents  $p(\tau)$  and the vector  $v' \in \mathbb{C}^{n+1}$  represents  $p'(\tau)$ .
- (ii) **(5 points)** What is the null space  $\mathcal{N}(D)$  and how does it relate to the derivatives of the polynomials?
- (iii) **(5 points)** Recall that the  $n+1$ -st derivative of a polynomial of degree less than or equal to  $n$  is identically 0. How is this reflected in the algebraic properties of  $D$ ?

## 2.b

(10 points)

Consider computing the matrix vector product  $y = Tx$ , i.e., you are given  $T$  and  $x$  and you want to compute  $y$ . Suppose further that the matrix  $T \in \mathbb{R}^{n \times n}$  is tridiagonal with constant values on each diagonal. For example, if  $n = 6$  then

$$\begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \alpha & \beta & 0 & 0 & 0 \\ 0 & \gamma & \alpha & \beta & 0 & 0 \\ 0 & 0 & \gamma & \alpha & \beta & 0 \\ 0 & 0 & 0 & \gamma & \alpha & \beta \\ 0 & 0 & 0 & 0 & \gamma & \alpha \end{pmatrix}$$

- (i) Write a simple loop-based psuedo-code that computes  $y = Tx$  for such a matrix  $T \in \mathbb{R}^n$ .
- (ii) How many operations are required as a function of  $n$ ?
- (iii) How many storage locations are required as a function of  $n$ ?



## Problem 3

(25 points)

### 3.a

(20 points)

Define the function  $f(x) = x - 1$  on the domain  $x > 1$ . Let  $x_0 \in \mathbb{R}$ ,  $x_0 > 2$ , and  $x_1 = x_0(1 + \delta)$  where  $\delta \in \mathbb{R}$  with  $|\delta| < 1$ .

- (i) (10 points) Determine the relative error between  $f(x_1)$  and  $f(x_0)$ , and the relative condition number  $\kappa_{rel}(x_0)$ .

- (ii) (10 points) Suppose  $|\delta| < 10^{-7}$ . Can we expect that the relative error between  $f(x_1)$  and  $f(x_0)$  is no more than  $10^{-4}$  for the region of values assumed for  $x_0$ ?



### 3.b

(5 points)

Suppose  $x$ ,  $y$  and  $z$  are floating point numbers in a standard model floating point arithmetic system. Is it true that

$$(x \boxed{op} (y \boxed{op} z)) = ((x \boxed{op} y) \boxed{op} z) ?$$

## Problem 4

(25 points)

### 4.a

(15 points)

If  $A \in \mathbb{R}^{n \times n}$  is a matrix with rank  $1 \leq k < n$  then there exists two matrices  $X \in \mathbb{R}^{n \times k}$  and  $Y \in \mathbb{R}^{n \times k}$  both of which have full column rank  $k$  and such that

$$A = XY^T$$

This is called a **full rank factorization** of  $A$ .

The reverse is also true, i.e., if there exist two matrices  $X \in \mathbb{R}^{n \times k}$  and  $Y \in \mathbb{R}^{n \times k}$  both of which have full column rank  $k$  such that  $A = XY^T$ , then the rank of  $A$  is  $k$ .

- (i) (5 points) Show that the full rank factorization of  $A$  is **not unique**.
- (ii) (5 points) Find a basis for  $\mathcal{R}(A)$ , the range of  $A$ .
- (iii) (5 points) Characterize a vector in the null space  $\mathcal{N}(A)$ .

## 4.b

**(10 points)**

Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric positive definite matrix. Suppose when computing the Cholesky factorization of  $A$  using IEEE floating point arithmetic we encounter at some step a computed Schur complement that is identically 0, i.e., every element in the computed Schur complement has the value of 0. What can we conclude about the original matrix  $A$ ? Justify your answer.

## Problem 5

(25 points)

### 5.a

(10 points)

Recall, that, given a full column rank matrix  $A \in \mathbb{R}^{n \times k}$ , we have discussed a reliable algorithm to compute an orthonormal basis of  $\mathcal{R}(A)$  by computing the Householder reflectors  $H_1^T, \dots, H_k^T$  that transform  $A$

$$H_k^T \dots H_1^T A = \begin{pmatrix} R \\ 0 \end{pmatrix}$$

where  $R$  is upper triangular and nonsingular, and then evaluating the computing efficiently the first  $k$  columns of

$$H = H_1 \dots H_k$$

to get the  $k$  orthonormal columns of  $Q$  where  $A = QR$ .

- (i) Suppose you want to compute an orthonormal basis of  $\mathcal{R}(A)$  but the  $H_i$  were not saved, i.e., we have only  $R$  and  $A$ . Describe how you would compute  $Q$  where  $A = QR$ . You need not worry about numerical issues.
- (ii) How many operations are required as a function of  $n$  and  $k$ ?

## 5.b

(15 points)

Suppose you are given the nonsingular tridiagonal matrix  $T \in \mathbb{R}^{n \times n}$ . For example, if  $n = 6$  then

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \gamma_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \\ 0 & \gamma_3 & \alpha_3 & \beta_3 & 0 & 0 \\ 0 & 0 & \gamma_4 & \alpha_4 & \beta_4 & 0 \\ 0 & 0 & 0 & \gamma_5 & \alpha_5 & \beta_5 \\ 0 & 0 & 0 & 0 & \gamma_6 & \alpha_6 \end{pmatrix}$$

- (i) Suppose you use Householder reflectors to transform  $A$  to upper triangular, i.e.,

$$H_{n-1}^T \dots H_1^T T = R.$$

What is the zero/nonzero structure of  $R$ ?

- (ii) What is the structure of each of the reflectors  $H_i$ ?
- (iii) What is the computational complexity of the factorization, i.e., what is  $k$  in  $O(n^k)$ ? (You do not have to determine the constant in the complexity expression.)

