

**Foundations of Computational Math I Exam 1**  
**Take-home Exam**  
**Open Notes, Textbook, Homework Solutions Only**  
**Calculators/Computers Allowed**  
**No collaborations with anyone**  
**Due beginning of Class Monday, December 5, 2011**

Question	Points Possible	Points Awarded
1. Linear Systems	30	
2. Linear Systems	30	
3. Nonlinear Equations	40	
4. Nonlinear Equations	30	
Total Points	130	

Name: \_\_\_\_\_

Alias: \_\_\_\_\_

to be used when posting anonymous grade list.

# Problem 1

(30 points)

Consider the system of equations

$$Ax = b$$

where  $A$  is a nonsingular lower triangular matrix, i.e.,

$$A = D - L$$

where  $D$  is diagonal and nonsingular, and  $L$  is strictly lower triangular.

(1.a) Show that Forward Gauss-Seidel will converge to  $x = A^{-1}b$  in a finite number of steps (in exact arithmetic) for any initial guess  $x_0$  and give a tight upper bound on the number of steps required.

(1.b) What is the complexity of solving the system using Forward Gauss-Seidel:  $O(n)$  or  $O(n^2)$  or  $O(n^3)$  or more?

(1.c) Show that Backward Gauss-Seidel will converge to  $x = A^{-1}b$  in a finite number of steps (in exact arithmetic) for any initial guess  $x_0$  and give a tight upper bound on the number of steps required or show that it does not converge in a finite number of steps for all  $x_0$ .

(1.d) What is the complexity of solving the system using Backward Gauss-Seidel:  $O(n)$  or  $O(n^2)$  or  $O(n^3)$  or more?

(1.e) What is the relationship between Backward Gauss-Seidel and Jacobi iterations for solving this system?

## Problem 2

(30 points)

### 2.a

(10 points)

(2.a.i) Suppose you must solve a linear system  $Ax = b$  and all you know about  $A$  is that it is nonsingular and well-conditioned. Suppose further you only have an iterative method that is applicable to linear systems where the coefficient matrix is symmetric positive definite. How would you solve  $Ax = b$ ?

(2.a.ii) When attempting to solve  $Ax = b$  where  $A$  is known to be nonsingular via an iterative method, we have seen various theorems that give sufficient conditions on  $A$  to guarantee the convergence of various iterative methods. It is not always easy to verify these conditions for a given matrix  $A$ . Let  $P$  and  $Q$  be two permutation matrices. Rather than solving  $Ax = b$  we could solve  $(PAQ)(Q^T x) = Pb$  using an iterative method. Sometimes it is possible to examine  $A$  and choose  $P$  and/or  $Q$  so that it is easy to apply one of our sufficient condition theorems.

Can you choose  $P$  and/or  $Q$  so that the permuted system converges for one or both of Gauss-Seidel and Jacobi with

$$A = \begin{pmatrix} 3 & 7 & -1 \\ 7 & 4 & 1 \\ -1 & 1 & 2 \end{pmatrix}?$$

## 2.b

(10 points)

Let

$$T = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}$$

and consider solving a linear system  $Tx = b$  with accelerated stationary Richardson's method

$$x_{k+1} = x_k + \alpha r_k$$

- (2.b.i) Can  $\alpha$  be set so that the iteration will converge for all  $x_0$ ? Justify your answer.
- (2.b.ii) If values of  $\alpha$  exist for which the iteration converges for all  $x_0$  how would you set its value so the convergence rate was optimal or nearly so? Give specific values in your response. If no such  $\alpha$  exists indicate what method you would use to solve the system.

## 2.c

(10 points)

Let

$$T = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & -1 & -4 & -1 \\ 0 & 0 & 0 & 0 & -1 & -4 \end{pmatrix}$$

and consider solving a linear system  $Tx = b$  with accelerated stationary Richardson's method

$$x_{k+1} = x_k + \alpha r_k$$

(2.c.i) Can  $\alpha$  be set so that the iteration will converge for all  $x_0$ ? Justify your answer.

(2.c.ii) If values of  $\alpha$  exist for which the iteration converges for all  $x_0$  how would you set its value so the convergence rate was optimal or nearly so? Give specific values in your response. If no such  $\alpha$  exists indicate what method you would use to solve the system.

## Problem 3

(40 points)

### 3.a

(30 points)

Let  $f(x) = x + \ln x$  on  $x > 0$  and define

$$\phi_1(x) = -\ln x$$

$$\phi_2(x) = e^{-x}$$

$$\phi_3(x) = \frac{\alpha x + e^{-x}}{\alpha + 1} \quad \alpha \neq -1$$

(3.a.i) Which of iteration functions  $\phi_1(x)$  and  $\phi_2(x)$  would you recommend to solve  $f(x) = 0$  on  $x > 0$ ? Justify your answer.

(3.a.ii) Suppose you have,  $\beta$ , an approximation to the root  $x_*$ , i.e.,  $\beta \approx x_*$ . How would you set the parameter  $\alpha$  in  $\phi_3(x)$  so that the resulting iteration was better than your choice between  $\phi_1(x)$  and  $\phi_2(x)$ ?

(3.a.iii) Show that the closer  $\beta$  is to the root  $x_*$  the faster the resulting convergence of  $\phi_3(x)$  using the associated  $\alpha$ .

(3.a.iv) Determine the iteration defined by Newton's method to solve  $f(x) = 0$  and comment on its convergence compared to  $\phi_3(x)$  using  $\beta$  and the associated  $\alpha$ .

(3.a.v) Use Newton's method and one of the others to determine the root  $x^*$ . You need not turn in any code simply report the computed value of  $x^*$  and indicate something about the observed rate of convergence.

### **3.b**

**(10 points)**

Discuss the main differences in convergence and complexity when solving a nonlinear equation using

(3.b.i) Newton's method

(3.b.ii) Secant method

(3.b.iii) Regula Falsi method

## Problem 4

(30 points)

Suppose you are asked to design an algorithm that could be used as an intrinsic library function to compute  $\sqrt[3]{\alpha}$  for any  $\alpha \in \mathbb{R}$  and  $\alpha > 0$ .

(4.a) Give an efficient algorithm in the form of  $x = \phi(x)$  that solves the problem.

(4.b) Determine the restrictions, if any, on your choice of  $x_0$ . Also indicate how you would generate  $x_0$  given  $\alpha$ .

(4.c) Determine the order of convergence for your method and the number of computations needed to get single precision accuracy given the initial accuracy for  $x_0$ .

(4.d) Use your algorithm to determine  $\sqrt[3]{69}$ . Give the value of  $x_0$  used and describe the observed convergence.